

A METRIC INTERPRETATION OF REFLEXIVITY FOR BANACH SPACES

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ABSTRACT. We define two metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$ on each Schreier family \mathcal{S}_α , $\alpha < \omega_1$, with which we prove the following metric characterization of reflexivity of a Banach space X : X is reflexive if and only if there is an $\alpha < \omega_1$, so that there is no mapping $\Phi : \mathcal{S}_\alpha \rightarrow X$ for which

$$cd_{\infty,\alpha}(A, B) \leq \|\Phi(A) - \Phi(B)\| \leq Cd_{1,\alpha}(A, B) \text{ for all } A, B \in \mathcal{S}_\alpha.$$

Secondly, we prove for separable and reflexive Banach spaces X , and certain countable ordinals α that $\max(\text{Sz}(X), \text{Sz}(X^*)) \leq \alpha$ if and only if $(\mathcal{S}_\alpha, d_{1,\alpha})$ does not bi-Lipschitzly embed into X . Here $\text{Sz}(Y)$ denotes the Szlenk index of a Banach space Y .

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we seek a metric characterization of reflexivity of Banach spaces. By a metric characterization of a property of a Banach space we mean a characterization which refers only to the metric structure of that space but not its linear structure. In 1976 Ribe [32] showed that two Banach spaces, which are uniformly homeomorphic, have uniformly linearly isomorphic finite-dimensional subspaces. In particular this means that the finite dimensional or *local* properties of a Banach space are determined by its metric structure. Based on this result Bourgain [11] suggested the “Ribe Program”, which asks to find metric descriptions of finite-dimensional invariants of Banach spaces. In [11] he proved the following characterization of super reflexivity: a Banach space X is super reflexive if and only if the binary trees B_n of length at most n , $n \in \mathbb{N}$, endowed with their graph metric, are not uniformly bi-Lipschitzly embedded into X . A binary tree of length at most n , is the set $B_n = \bigcup_{k=0}^n \{-1, 1\}^k$, with the *graph* or *shortest path metric* $d(\sigma, \sigma') = i + j - 2 \max\{t \geq 0 : \sigma_s = \sigma'_s : s = 1, 2, \dots, t\}$, for $\sigma = (\sigma_s)_{s=1}^i \neq \sigma' = (\sigma'_s)_{s=1}^j$ in $\bigcup_{k=0}^n \{-1, 1\}^k$. A new and shorter proof of this result was recently obtained by Kloeckner in [19]. In [7] Baudier extended this result and proved that a Banach space X is super reflexive, if and only if the infinite binary tree $B_\infty = \bigcup_{n=0}^\infty \{-1, 1\}^n$ (with the graph distance) does not bi-Lipschitzly embed into X . Nowadays this result can be deduced from Bourgain’s result and a result of Ostrovskii’s [28, Theorem 1.2] which states that a locally finite metric space A embeds bi-Lipschitzly into a Banach space X if all of its finite subsets uniformly bi-Lipschitzly embed into X . In [18] Johnson and Schechtman characterized superflexivity, using the *Diamond Graphs*, D_n , $n \in \mathbb{N}$, and proved that Banach space X is super reflexive if and only if the D_n , $n \in \mathbb{N}$, do not uniformly bi-Lipschitzly embed into X . There are several other local properties, *i.e.*, properties of the finite dimensional subspaces of Banach spaces, for which metric characterizations were found. The following are some examples: Bourgain, Milman and Wolfson [12] characterized having non trivial type using *Hammond cubes* (the sets B_n , together with the ℓ_1 -norm), and Mendel and Naor [21, 22] present metric characterizations of Banach spaces with type p , $1 < p \leq 2$, and cotype q , $2 \leq q < \infty$. For a more extensive account on the Ribe program we would like refer the reader to the survey articles [6, 23] and the book [29].

Instead of only asking for metric characterizations of local properties, one can also ask for metric characterizations of other properties of Banach space, properties which might not be determined by the finite dimensional subspaces. A result in this direction was obtained by Baudier, Kalton and Lancien in [9]. They showed that a reflexive Banach space X has a renorming, which is *asymptotically uniformly convex* (AUC) and *asymptotically uniformly smooth* (AUS), if and only if the countably branching trees of length $n \in \mathbb{N}$, T_n do not uniformly bi-Lipschitzly embed into X . Here $T_n = \bigcup_{k=0}^n \mathbb{N}^k$, together with the graph metric, *i.e.*, $d(a, b) = i + j - \max\{t \geq 0 : a_s = b_s, s = 1, 2, \dots, t\}$, for $a = (a_1, a_2, \dots, a_i) \neq b = (b_1, b_2, \dots, b_j)$ in T^n . Among the many equivalent conditions for a reflexive Banach space X to be (AUC)- and (AUS)-renormable (see [25]) one of them states that $Sz(X) = Sz(X^*) = \omega$, where $Sz(Z)$ denotes the *Szlenk index* of a Banach space Z (see Section 5 for the definition and properties of the Szlenk index). In [13] Dilworth, Kutzarova, Lancien and Randrianarivony, showed that a separable Banach space X is reflexive and (AUC)- and (AUS)-renormable if and only if X admits an equivalent norm for which X has *Rolewicz’s β -property*. According to [20] a

Banach space X has Rolewicz's β -property if and only if

$$\bar{\beta}_X(t) = 1 - \sup \left\{ \inf \left\{ \frac{\|x - x_n\|}{2} : n \geq 1 \right\} : (x_n)_{n=1}^\infty \subset B_X, \text{sep}[(x_n)] \geq t, x \in B_X \right\} > 0,$$

for all $t > 0$, where $\text{sep}[(z_n)] = \inf_{m \neq n} \|z_m - z_n\|$, for a sequence $(z_n) \subset X$. The function $\bar{\beta}_X$ is called the β -modulus of X . Using the equivalence between the positivity of the β -modulus and the property that a separable Banach space is reflexive and (AUC)- and (AUS)-renormable, Baudier [8] was able to establish a new and shorter proof of the above cited result from [9]. Metric descriptions of other non-local Banach space properties, for example the Radon-Nikodým Property, can be found in [30].

In our paper we would like to concentrate on metric descriptions of the property that a Banach space is reflexive, and subclasses of reflexive Banach spaces. In [30], Ostrovskii established a *submetric* characterization of reflexivity. Let T be the set of all pairs (x, y) in $\ell_1 \times \ell_1$, for which $\|x - y\|_1 \leq 2\|x - y\|_s$, where $\|\cdot\|_1$ denotes the usual norm on ℓ_1 and $\|\cdot\|_s$ denotes the *summing norm*, i.e., $\|z\|_s = \sup_{k \in \mathbb{N}} \left| \sum_{j=1}^k z_j \right|$, for $z = (z_j) \in \ell_1$. [30, Theorem 3.1] states that a Banach space X is not reflexive if and only if there is a map $f : \ell_1 \rightarrow X$ and a number $0 < c \leq 1$, so that $c\|x - y\|_1 \leq \|f(x) - f(y)\| \leq \|x - y\|_1$ for all $(x, y) \in T$. In Section 12 we will formulate a similar result, using a discrete subset of $\ell_1 \times \ell_1$, witnessing the same phenomena. Recently Procházka [31, Theorem 3] formulated an interesting metric description of reflexivity. He constructed a uniformly discrete metric space M_R with the following properties: If M_R bi-Lipschitzly embeds into a Banach space X with distortion less than 2, then X is non reflexive. The *distortion* of a bi-Lipschitz embedding f of one metric into another is the product of the Lipschitz constant of f and the Lipschitz constant of f^{-1} . Conversely, if X is non reflexive, then there exists a renorming $|\cdot|$ of X , so that M_R embeds into $(X, |\cdot|)$ isometrically.

Our paper has the goal to find a metric characterization of reflexivity. An optimal result would be a statement, similar to Bourgain's result, of the form "all members of a certain family $(M_i)_{i \in I}$ of metric spaces embed uniformly bi-Lipschitzly into a space X if and only if X is not reflexive". In the language, introduced by Ostrovskii [25], this would mean that $(M_i)_{i \in I}$ is a *family of test spaces for reflexivity*. Instead, our result will be of the form (see Theorem A below), "there is a family of sets $(M_i)_{i \in I}$, and for $i \in I$, there are metrics $d_{\infty, i}$ and $d_{1, i}$ on M_i , with the property, that a given space X is non reflexive if and only if there are injections $\Phi_i : M_i \rightarrow X$ and $0 < c \leq 1$ so that $cd_{\infty, i}(x, y) \leq \|x - y\| \leq d_{1, i}(x, y)$, for all $x, y \in M_i$ ". In Section 12 we will discuss the difficulties, in arriving to a characterization of reflexivity of the first form. Nevertheless, if we restrict ourselves to the class of reflexive spaces we arrive to a metric characterization for the *complexity* of a given space, which we measure by the Szlenk index, using test spaces. Roughly speaking, the higher the Szlenk index is of a given Banach space, the more averages of a given weakly null sequence one has to take to arrive to a norm null sequence. For a precise formulation of this statement we refer to Theorem 5.3. For the class of separable and reflexive spaces we will introduce an uncountable family of metric spaces $(M_\alpha)_{\alpha < \omega_1}$ for which we will show that the higher the complexity of a given reflexive and separable space X or its dual X^* is, the more members of $(M_\alpha)_{\alpha < \omega_1}$ can be uniformly bi-Lipschitzly embedded into X .

The statements of our main results are as follows. The definition of the *Schreier families* $\mathcal{S}_\alpha \subset [\mathbb{N}]^{<\omega}$, for $\alpha < \omega_1$, will be recalled in Section 2, the *Szlenk index* $\text{Sz}(X)$ for a Banach space X in Section 5, and the two metrics $d_{1, \alpha}$ and $d_{\infty, \alpha}$ on \mathcal{S}_α will be defined in Section 7. The statements of our main results are as follows.

Theorem A. A separable Banach space X is reflexive if and only if there is an $\alpha < \omega_1$ for which there does not exist a map $\Phi : \mathcal{S}_\alpha \rightarrow X$, with the property that for some numbers $C \geq c > 0$

$$(1) \quad cd_{\infty,\alpha}(A, B) \leq \|\Phi(A) - \Phi(B)\| \leq Cd_{1,\alpha}(A, B) \text{ for all } A, B \in \mathcal{S}_\alpha.$$

Definition 1.1. Assume that X is a Banach space, $\alpha < \omega_1$, and $C \geq c > 0$. We call a map $\Phi : \mathcal{S}_\alpha \rightarrow X$, with the property that for all $A, B \in \mathcal{S}_\alpha$,

$$(2) \quad cd_{\infty,\alpha}(A, B) \leq \|\Phi(A) - \Phi(B)\| \leq Cd_{1,\alpha}(A, B)$$

a c -lower- $d_{\alpha,\infty}$ and C -upper- $d_{\alpha,1}$ embedding of \mathcal{S}_α into X . If \mathcal{A} is a subset of \mathcal{S}_α , and $\Phi : \mathcal{A} \rightarrow X$, is a map which satisfies (2) for all $A, B \in \mathcal{A}$, we call it a c -lower- $d_{\alpha,\infty}$ and C -upper- $d_{\alpha,1}$ embedding of \mathcal{A} into X .

Our next result extends one direction (the “easy direction”) of the main result of [9] to spaces with higher order Szlenk indices. As in [9] reflexivity is not needed here.

Theorem B. Assume that X is a separable Banach space and that $\max(\text{Sz}(X), \text{Sz}(X^*)) > \omega^\alpha$, for some countable ordinal α . Then $(\mathcal{S}_\alpha, d_{1,\alpha})$ embeds bi-Lipschitzly into X and X^* .

We will deduce one direction of Theorem A from James’s characterization of reflexive Banach spaces [16], and show that for any non reflexive Banach space X and any $\alpha < \omega_1$ there is a map $\Phi_\alpha : \mathcal{S}_\alpha \rightarrow X$ which satisfies (1). The converse will follow from the following result.

Theorem C. Assume that X is a reflexive and separable Banach space. Let $\xi < \omega_1$ and put $\beta = \omega^{\omega^\xi}$. If for some numbers $C > c > 0$ there exists a c -lower- $d_{\alpha,\infty}$ and C -upper- $d_{\alpha,1}$ embedding of \mathcal{S}_{β^2} into X , then $\text{Sz}(X) > \omega^\beta$ or $\text{Sz}(X^*) > \beta$.

Theorem C, and thus the missing part of Theorem A, will be shown in Section 11, Theorem 11.6. Combining Theorems B and C, we obtain the following characterization of certain bounds of the Szlenk index of X and its dual X^* . This result extends the main result of [9] to separable and reflexive Banach spaces with higher order Szlenk indices.

Corollary 1.2. Assume that $\omega < \alpha < \omega_1$ is an ordinal for which $\omega^\alpha = \alpha$. Then the following statements are equivalent for a separable and reflexive space X

- a) $\max(\text{Sz}(X), \text{Sz}(X^*)) \leq \alpha$,
- b) $(\mathcal{S}_\alpha, d_1)$ is not bi-Lipschitzly embeddable into X .

Corollary 1.2 and a result in [26] yield the following corollary. We thank Christian Rosendal who pointed it out to us.

Corollary 1.3. If $\alpha < \omega_1$, with $\alpha = \omega^\alpha$, the class of all Banach spaces X for which $\max(\text{Sz}(X), \text{Sz}(X^*)) \leq \alpha$ is Borel in the Effros-Borel structure of closed subspaces of $C[0, 1]$.

A proof of Corollaries 1.2 and 1.3 will be presented at the end of Section 11. For the proof of our main results we will need to introduce some notation and to make several preliminary observations. The reader who is at first only interested to understand our main results will only need the definition of the *Schreier families* \mathcal{S}_α , $\alpha < \omega_1$, given in Subsection 2.2, the definition of *repeated averages* stated in the beginning of Section 3, and the definition of the two metrics $d_{1,\alpha}$ and $d_{\infty,\alpha}$ on \mathcal{S}_α introduced in Section 7.

2. REGULAR FAMILIES, SCHREIER FAMILIES AND FINE SCHREIER FAMILIES

In this section we will first recall the definition of general *Regular Subfamilies of* $[\mathbb{N}]^{<\omega}$. Then we recall the definition of the *Schreier Families* \mathcal{S}_α and the *Fine Schreier Families* $\mathcal{F}_{\beta,\alpha}$, $\alpha \leq \beta < \omega_1$ [1]. The recursive definition of both families depends on choosing for every limit ordinal a sequence (α_n) , which increases to α . In order to ensure that later our proof will work out, we will need (α_n) to satisfy certain conditions.

2.1. Regular Families in $[\mathbb{N}]^{<\omega}$. For a set S we denote the subsets, the finite subsets, and the countably infinite subsets by $[S]$, $[S]^{<\omega}$, and $[S]^\omega$. We always write subsets of \mathbb{N} in increasing order. Thus, if we write $A = \{a_1, a_2, \dots, a_n\} \in [\mathbb{N}]^{<\omega}$, or $A = \{a_1, a_2, \dots\} \in [\mathbb{N}]^\omega$ we always assume that $a_1 < a_2 < \dots$. Identifying the elements of $[\mathbb{N}]$ in the usual way with elements of $\{0, 1\}^\omega$, we consider on $[\mathbb{N}]$ the product topology of the discrete topology on $\{0, 1\}$. Note that it follows for a sequence $(A_n) \subset [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^{<\omega}$, that (A_n) converges to A if and only if for all $k \geq \max A$ there is an m so that $A_n \cap [1, k] = A$, for all $n \geq m$.

For $A \in [\mathbb{N}]^{<\omega}$ and $B \in [\mathbb{N}]$ we write $A < B$, if $\max(A) < \min(B)$. As a matter of convention we put $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$, and thus $A < \emptyset$ and $\emptyset > A$ is true for all $A \in [\mathbb{N}]^{<\omega}$. For $m \in \mathbb{N}$ we write $m \leq A$ or $m < A$, if $m \leq \min(A)$, or $m < \min(A)$, respectively.

We denote by \preceq the partial order of *extension* on $[\mathbb{N}]^{<\omega}$, i.e., $A = \{a_1, a_2, \dots, a_l\} \preceq B = \{b_1, b_2, \dots, b_m\}$ if $l \leq m$ and $a_i = b_i$, for $i = 1, 2, \dots, l$, and we write $A \prec B$ if $A \preceq B$ and $A \neq B$.

We say that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *closed under taking restrictions* or *is a subtree of* $[\mathbb{N}]^{<\omega}$ if $A \in \mathcal{F}$ whenever $A \prec B$ and $B \in \mathcal{F}$, *hereditary* if $A \in \mathcal{F}$ whenever $A \subset B$ and $B \in \mathcal{F}$, and \mathcal{F} is called *compact* if it is compact in the product topology. Note that a family which is closed under restrictions is compact if and only if it is *well founded*, i.e., if it does not contain strictly ascending chains with respect to extensions. Given n , $a_1 < \dots < a_n$, $b_1 < \dots < b_n$ in \mathbb{N} we say that $\{b_1, \dots, b_n\}$ is a *spread* of $\{a_1, \dots, a_n\}$ if $a_i \leq b_i$ for $i = 1, \dots, n$. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called *spreading* if every spread of every element of \mathcal{F} is also in \mathcal{F} . We sometimes have to pass from a family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ to the subfamily $\mathcal{F} \cap [N]^\omega = \{A \in \mathcal{F} : A \subset N\}$, where $N \subset \mathbb{N}$ is infinite. Of course we might lose the property that \mathcal{F} is spreading. Nevertheless $\mathcal{F} \cap [N]^\omega$ will be called *spreading relative to N* , if with $A \in \mathcal{F} \cap [N]^{<\omega}$ every spread of A which is a subset of N is in \mathcal{F} . Note that if $\mathcal{F} \subset [N]^{<\omega}$ is spreading relatively to $N = \{n_1, n_2, \dots\} \in [\mathbb{N}]^\omega$, then

$$\tilde{\mathcal{F}} = \{A \in [\mathbb{N}]^{<\omega} : \{n_j : j \in A\} \in \mathcal{F}\},$$

is spreading.

A second way to pass to a sub families is the following. Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ and $N = \{n_1, n_2, \dots\} \in [\mathbb{N}]^\omega$, then we call

$$\mathcal{F}^N = \{\{n_j : j \in A\} : A \in \mathcal{F}\},$$

the *spread of \mathcal{F} onto N* .

A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called *regular* if it is hereditary, compact, and spreading. Note that if $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact, spreading and closed under restriction it is also hereditary and thus regular. Indeed if $B = \{b_1, b_2, \dots, b_l\} \in \mathcal{F}$ and $1 \leq i_1 < i_2 < \dots < i_k \leq l$, then $A = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\}$ is a spread of $B' = \{b_1, b_2, \dots, b_k\}$ and since $B' \in \mathcal{F}$, it also follows that $A \in \mathcal{F}$.

If $\mathcal{F} \subset [N]^{<\omega}$, for some infinite set $N \subset \mathbb{N}$, is called *regular relatively to N* , if it is compact, spreading relatively to N and hereditary.

If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ we denote the maximal elements of \mathcal{F} , *i.e.*, the elements $A \in \mathcal{F}$ for which there is no $B \in \mathcal{F}$ with $A \prec B$, by $\text{MAX}(\mathcal{F})$. Note that if \mathcal{F} is compact every element in \mathcal{F} can be extended to a maximal element in \mathcal{F} .

For $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^{<\omega}$ we define

$$\mathcal{F}(A) = \{B \in [\mathbb{N}]^{<\omega} : A < B, A \cup B \in \mathcal{F}\}.$$

Note that if \mathcal{F} is compact, spreading, closed under restrictions, or hereditary, so is $\mathcal{F}(A)$.

If $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact, we denote by $\text{CB}(\mathcal{F})$ its *Cantor Bendixson* index, which is defined as follows. We first define the *derivative* of \mathcal{F} by

$$\begin{aligned} \mathcal{F}' &= \left\{ A \in \mathcal{F} : \exists (A_n) \subset \mathcal{F} \setminus \{A\} \quad A_n \rightarrow_{n \rightarrow \infty} A \right\} \\ &= \left\{ A \in \mathcal{F} : \exists (B_n) \subset \mathcal{F}(A) \setminus \{\emptyset\} \quad B_n \rightarrow_{n \rightarrow \infty} \emptyset \right\} = \mathcal{F} \setminus \{A \in \mathcal{F} : A \text{ is isolated in } \mathcal{F}\}. \end{aligned}$$

Note that if \mathcal{F} is hereditary then \mathcal{F}' is hereditary, but also note that if \mathcal{F} is only closed under restrictions, this is not necessarily true for \mathcal{F}' . Indeed, consider for example

$$\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, n\}, n > 2\}.$$

Every maximal element A of \mathcal{F} is not in \mathcal{F}' and if \mathcal{F} is spreading, then

$$\mathcal{F}' = \mathcal{F} \setminus \text{MAX}(\mathcal{F}).$$

For $A \in [\mathbb{N}]^{<\omega}$ it follows that

$$(3) \quad \mathcal{F}'(A) = (\mathcal{F}(A))'.$$

Indeed

$$\begin{aligned} B \in \mathcal{F}'(A) &\iff B > A \text{ and } A \cup B \in \mathcal{F}' \\ &\iff B > A \text{ and } \exists (C_n) \in \mathcal{F} \setminus \{A \cup B\} \quad C_n \rightarrow_{n \rightarrow \infty} A \cup B \\ &\iff B > A \text{ and } \exists (B_n) \in \mathcal{F}(A) \setminus \{B\} \quad B_n \rightarrow_{n \rightarrow \infty} B \iff B \in (\mathcal{F}(A))' \\ &\quad [\text{Choose } B_n = C_n \setminus A, \text{ for } n \text{ large enough}]. \end{aligned}$$

By transfinite induction we define for each ordinal α the α -th derivative of \mathcal{F} , by

$$\mathcal{F}^{(0)} = \mathcal{F}, \mathcal{F}^{(\alpha)} = (\mathcal{F}^{(\gamma)})', \text{ if } \alpha = \gamma + 1, \text{ and } \mathcal{F}^{(\alpha)} = \bigcap_{\gamma < \alpha} \mathcal{F}^{(\gamma)}, \text{ if } \alpha \text{ is a limit ordinal.}$$

It follows that $\mathcal{F}^{(\beta)} \subset \mathcal{F}^{(\alpha)}$ if $\alpha \leq \beta$. By transfinite induction (3) generalizes to

$$(4) \quad \mathcal{F}^{(\alpha)}(A) = (\mathcal{F}(A))^{(\alpha)}, \text{ for all } A \in [\mathbb{N}]^{<\omega} \text{ and ordinal } \alpha.$$

Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact. Since \mathcal{F} is countable and since every countable and compact metric space has isolated points, it follows that for some $\alpha < \omega_1$ the α -th derivative of \mathcal{F} is empty and we define

$$\text{CB}(\mathcal{F}) = \min\{\alpha : \mathcal{F}^{(\alpha)} = \emptyset\}.$$

$\text{CB}(\mathcal{F})$ is always a successor ordinal. Indeed, if $\alpha < \omega_1$ is a limit ordinal and $\mathcal{F}^{(\gamma)} \neq \emptyset$ for all $\gamma < \alpha$, it follows that $\mathcal{F}^{(\alpha)} = \bigcap_{\gamma < \alpha} \mathcal{F}^{(\gamma)} \neq \emptyset$.

Definition 2.1. For $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]$ we define

$$(5) \quad \mathcal{F} \sqcup_{<} \mathcal{G} := \{A \cup B : A \in \mathcal{F}, B \in \mathcal{G} \text{ and } A < B\}$$

$$(6) \quad \mathcal{F}[\mathcal{G}] := \left\{ \bigcup_{i=1}^n B_i : n \in \mathbb{N}, B_1 < B_2 < \dots < B_n, B_i \in \mathcal{G}, i = 1, 2, \dots, n, \right. \\ \left. \text{and } \{\min(B_i) : i = 1, 2, \dots, n\} \in \mathcal{F} \right\}.$$

It is not hard to see that if \mathcal{F} and \mathcal{G} are regular families so are $\mathcal{F} \sqcup_{<} \mathcal{G}$ and $\mathcal{F}[\mathcal{G}]$.

2.2. The Schreier families. We define the *Schreier families* $\mathcal{S}_\alpha \subset [\mathbb{N}]^{<\omega}$ by transfinite induction for all $\alpha < \omega_1$, as follows :

$$(7) \quad \mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$$

if $\alpha = \gamma + 1$, we let

$$(8) \quad \mathcal{S}_\alpha = \mathcal{S}_1[\mathcal{S}_\gamma] = \left\{ \bigcup_{j=1}^n E_j : n \leq \min(E_1), E_1 < E_2 < \dots < E_n, E_j \in \mathcal{S}_\gamma, j = 1, 2, \dots, n \right\},$$

and if α is a limit ordinal we choose a fixed sequence $(\lambda(\alpha, n) : n \in \mathbb{N}) \subset [1, \alpha)$ which increases to α and put

$$(9) \quad \mathcal{S}_\alpha = \{E : \exists k \leq \min(E), \text{ with } E \in \mathcal{S}_{\lambda(\alpha, k)}\}.$$

An easy induction shows that \mathcal{S}_α is a hereditary, compact and spreading family for all $\alpha < \omega_1$. It is not very hard to see by transfinite induction that \mathcal{S}_α is in the following very limited sense *backwards spreading*:

$$(10) \quad \text{If } A = \{a_1, a_2, \dots, a_n\} \in \mathcal{S}_\alpha, \text{ then } \{a_1, a_2, \dots, a_{n-1}, a_n - 1\} \in \mathcal{S}_\alpha.$$

So, in particular, if $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ is not maximal, then $(A \cup \{k\})_{k > \max(A)} \subset \mathcal{S}_\alpha$.

Secondly, by transfinite induction we can easily prove that \mathcal{S}_α is “almost” increasing in α , in the following sense:

Proposition 2.2. *For all ordinals $\alpha < \beta < \omega$, there is an $n \in \mathbb{N}$ so that*

$$\mathcal{S}_\alpha \cap [n, \infty)^{<\omega} \subset \mathcal{S}_\beta.$$

The following formula for $\text{CB}(\mathcal{S}_\alpha)$ is well know and can easily be shown by transfinite induction for all $\alpha < \omega_1$.

Proposition 2.3. *For $\alpha < \omega_1$ we have $\text{CB}(\mathcal{S}_\alpha) = \omega^\alpha + 1$.*

We now make further assumptions on the approximating sequence $(\lambda(\alpha, n)) \subset [1, \alpha)$, we had chosen to define the Schreier family \mathcal{S}_α , for limit ordinals $\alpha < \omega_1$. And we will choose $(\lambda(\alpha, n))$ recursively. Assume that α is a countable limit ordinal and that we have defined $(\lambda(\gamma, n))$, for all limit ordinals $\gamma < \alpha$, and thus, \mathcal{S}_γ for all $\gamma < \alpha$.

Recall that α , can be represented uniquely in its *Cantor Normal Form*

$$(11) \quad \alpha = \omega^{\xi_k} m_k + \omega^{\xi_{k-1}} m_{k-1} + \dots + \omega^{\xi_1} m_1,$$

where $\xi_k > \xi_{k-1} > \dots > \xi_1$, $m_k, m_{k-1}, \dots, m_1 \in \mathbb{N}$, and, since α is a limit ordinal, $\xi_1 \geq 1$.

We distinguish between three cases:

Case 1. $k \geq 2$ or $m_1 \geq 2$. In that case we put for $n \in \mathbb{N}$

$$(12) \quad \lambda(\alpha, n) = \omega^{\xi_k} m_k + \omega^{\xi_{k-1}} m_{k-1} + \dots + \omega^{\xi_1} (m_1 - 1) + \lambda(\omega^{\xi_1}, n)$$

Before considering the next cases we need to make the following observation:

Proposition 2.4. *Assume that for all limit ordinals $\gamma \leq \alpha$ satisfying Case 1 the approximating sequences $(\lambda(\gamma, n) : n \in \mathbb{N})$ satisfies the above condition (12). It follows for all $\gamma \leq \alpha$, with*

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} m_1,$$

being the Cantor Normal Form, that

$$(13) \quad \mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}], \text{ where for some } j = 1, \dots, l$$

$$\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_j} m_j^{(1)} \text{ and } \gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1,$$

$$\text{with } m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}, m_j = m_j^{(1)} + m_j^{(2)}.$$

Proof. We will show (13) by transfinite induction for all $\gamma \leq \alpha$. Assume that (13) holds for all $\tilde{\gamma} < \gamma$. If $\gamma = \omega^\xi$, then (13) is trivially satisfied. Indeed, in that case $\gamma = \gamma + 0 = 0 + \gamma$, are the only two choices to write γ as the sum of two ordinals, and we observe that $\mathcal{S}_0[\mathcal{S}_\gamma] = \mathcal{S}_\gamma[\mathcal{S}_0] = \mathcal{S}_\gamma$.

It is left to verify (13) in the case that $l \geq 2$ or $m_l \geq 2$. Let $\gamma = \gamma_1 + \gamma_2$ be a decomposition of γ as in the statement of (13). We can without loss of generality assume that $\gamma_2 > 0$.

If $\gamma_2 = \beta + 1$ for some β (which implies that γ itself is a successor ordinal) it follows from the induction hypothesis and (8) that $\mathcal{S}_{\gamma_1+\beta+1} = \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$, so we need to show that

$$\mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]] = \mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}].$$

If $A \in \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$, we can write A as $A = \bigcup_{i=1}^m A_i$ with $m \leq A_1 < A_2 < \dots < A_m$ and $A_i \in \mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]$, for $i = 1, \dots, m$ which in turn means that $A_i = \bigcup_{j=1}^{m_i} A_{(i,j)}$, where $A_{(i,1)} < A_{(i,2)} < \dots < A_{(i,l_i)}$, $A_{(i,j)} \in \mathcal{S}_{\gamma_1}$, for $j = 1, 2, \dots, l_i$, and $\{\min(A_{(i,j)} : j = 1, 2, \dots, l_i)\} \in \mathcal{S}_\beta$, for $i = 1, 2, \dots, m$. This means that $\{\min A_{(i,j)} : j = 1, 2, \dots, l_i, i = 1, 2, \dots, m\}$ is in $\mathcal{S}_{\beta+1}$ and thus we conclude that $A \in \mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}]$. Conversely, we can show in a similar way that $\mathcal{S}_{\beta+1}[\mathcal{S}_{\gamma_1}] \subset \mathcal{S}_1[\mathcal{S}_\beta[\mathcal{S}_{\gamma_1}]]$.

If γ_2 is a limit ordinal we first observe that

$$\lambda(\gamma, n) = \lambda(\gamma_1 + \gamma_2, n) = \gamma_1 + \lambda(\gamma_2, n).$$

If $A \in \mathcal{S}_{\gamma_1+\gamma_2}$ it follows that there is an $n \leq \min A$ so that, using the induction hypothesis, we have

$$A \in \mathcal{S}_{\gamma_1+\lambda(\gamma_2, n)} = \mathcal{S}_{\lambda(\gamma_2, n)}[\mathcal{S}_{\gamma_1}].$$

This means that $A = \bigcup_{j=1}^m A_j$ with, $A_1 < A_2 < \dots < A_m$, $\{\min(A_j) : j = 1, 2, \dots, m\} \in \mathcal{S}_{\lambda(\gamma_2, n)}$ and $A_j \in \mathcal{S}_{\gamma_1}$, for $j = 1, 2, \dots, m$. Since $n \leq \min(A) = \min(A_1)$, it follows that $\{\min(A_j) : j = 1, 2, \dots, m\} \in \mathcal{S}_{\gamma_2}$, and, thus, that $A \in \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}]$. Conversely, we can similarly show that if $A \in \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}]$, then it follows that $A \in \mathcal{S}_{\gamma_1+\gamma_2}$. \square

If Case 1 does not hold α must be of the form $\alpha = \omega^\gamma$.

Case 2. $\alpha = \omega^{\omega^\kappa}$, for some $\kappa < \omega_1$. In that cases we make the following requirement on the sequence $(\lambda(\alpha, n) : n \in \mathbb{N})$:

$$(14) \quad \mathcal{S}_{\lambda(\alpha, n)} \subset \mathcal{S}_{\lambda(\alpha, n+1)}, \text{ for all } n \in \mathbb{N}.$$

We can assure (14) as follows: first choose any sequence $\lambda'(\alpha, n)$, which increases to α . Then we notice that Proposition 2.2 yields that for a fast enough increasing sequence $(l_n) \subset \mathbb{N}$, it follows that $\mathcal{S}_{\lambda'(\alpha, n)+l_n} \subset \mathcal{S}_{\lambda'(\alpha, n+1)+l_{n+1}}$. Indeed, we first note that the only set $A \in \mathcal{S}_\gamma$, $\gamma < \alpha$ which contains 1, must be the singleton $A = \{1\}$. This follows easily by induction.

Secondly we note that by (8) it follows that $\{2, 3, \dots, n\} \subset \mathcal{S}_{\gamma+n}$, for each $\gamma < \alpha$, and $n \in \mathbb{N}$, which yields our claim.

The remaining case is the following

Case 3. $\alpha = \omega^{\omega^\kappa + \xi}$, where $1 \leq \xi \leq \omega^\kappa$.

We first observe that in this case κ and ξ are uniquely defined.

Lemma 2.5. *Let α be an ordinal number so that there are ordinal numbers κ, ξ with $\xi \leq \omega^\kappa$ and $\alpha = \omega^{\omega^\kappa + \xi}$. Then for every κ', ξ' with $\xi' \leq \omega^{\kappa'}$ so that $\alpha = \omega^{\omega^{\kappa'} + \xi'}$, we have $\kappa = \kappa'$ and $\xi = \xi'$.*

Proof. Let $\alpha = \omega^{\omega^\kappa + \xi} = \omega^{\omega^{\kappa'} + \xi'}$ be as above. By [34, Theorem 41, §7.2] $\omega^\kappa + \xi = \omega^{\kappa'} + \xi'$. If $\kappa' < \kappa$, then $\omega^{\kappa'} + \xi' \leq \omega^{\kappa'} 2 < \omega^{\kappa'} \omega = \omega^{\kappa'+1} \leq \omega^\kappa \leq \omega^\kappa + \xi$, which is a contradiction. We conclude that $\kappa \leq \kappa'$, and therefore by interchanging the roles of κ and κ' we obtain that $\kappa = \kappa'$. In conclusion, $\omega^\kappa + \xi = \omega^{\kappa'} + \xi'$ and therefore $\xi = \xi'$ as well. \square

We choose now a sequence $(\theta(\xi, n))_n$ of ordinal numbers increasing to ω^ξ , so that

$$(15) \quad \mathcal{S}_{\omega^{\omega^\kappa} \theta(\xi, n)} \subset \mathcal{S}_{\omega^{\omega^\kappa} \theta(\xi, n+1)}$$

and define

$$(16) \quad \lambda(\alpha, n) = \omega^{\omega^\kappa} \theta(\xi, n).$$

We describe how (15) can be obtained. Start with an arbitrary sequence $(\theta'(\xi, n))_n$ increasing to ω^ξ . We shall recursively choose natural numbers $(k_n)_{n \in \mathbb{N}}$, so that setting $\theta(\xi, n) = \theta'(\xi, n) + k_n$, (15) is satisfied. Assuming that k_1, \dots, k_n have been chosen choose k_{n+1} , as in the argument yielding (14), so that

$$\mathcal{S}_{\omega^{\omega^\kappa} \theta(\xi, n)} \subset \mathcal{S}_{\omega^{\omega^\kappa} \theta'(\xi, n+1) + k_{n+1}}.$$

We will show that this k_{n+1} is the desired natural number, i.e. that

$$\mathcal{S}_{\omega^{\omega^\kappa} \theta(\xi, n)} \subset \mathcal{S}_{\omega^{\omega^\kappa} (\theta'(\xi, n+1) + k_{n+1})}.$$

First note that using finite induction and Proposition (2.4), it is easy to verify that for $\gamma < \alpha$, with $\gamma = \omega^\xi$, for some $\xi < \omega_1$, and $n \in \mathbb{N}$

$$(17) \quad \mathcal{S}_{\gamma \cdot n} = \underbrace{\mathcal{S}_\gamma [\mathcal{S}_\gamma \cdots \mathcal{S}_\gamma [\mathcal{S}_\gamma]]}_{n\text{-times}}.$$

and thus

$$\begin{aligned} \mathcal{S}_{\omega^{\omega^\kappa} (\theta'(\xi, n+1) + k_{n+1})} &= \mathcal{S}_{\omega^{\omega^\kappa} \theta'(\xi, n+1) + \omega^{\omega^\kappa} k_{n+1}} = \underbrace{\mathcal{S}_{\omega^{\omega^\kappa}} [\cdots [\mathcal{S}_{\omega^{\omega^\kappa}} [\mathcal{S}_{\omega^{\omega^\kappa} \theta'(\xi, n+1)}]]]}_{k_{n+1}\text{-times}} \\ &\supset \underbrace{\mathcal{S}_1 [\cdots [\mathcal{S}_1 [\mathcal{S}_{\omega^{\omega^\kappa} \theta'(\xi, n+1)}]]]}_{k_{n+1}\text{-times}} = \mathcal{S}_{\omega^{\omega^\kappa} \theta'(\xi, n+1) + k_{n+1}} \supset \mathcal{S}_{\omega^{\omega^\kappa} \theta(\xi, n)}. \end{aligned}$$

We point out that the sequence $(\theta(\xi, n))_n$ also depends on α .

Proposition 2.6. *Assuming the approximating sequences $(\lambda(\alpha, n) : n \in \mathbb{N})$ satisfy for all limit ordinals α the above conditions. It follows for all $\gamma < \omega_1$, with*

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} m_1,$$

being the Cantor Normal Form, that

$$(18) \quad \mathcal{S}_{\lambda(\gamma, n)} \subset \mathcal{S}_{\lambda(\gamma, n+1)} \text{ for all } n \in \mathbb{N}, \text{ if } \gamma \text{ is a limit ordinal.}$$

$$(19) \quad \mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}[\mathcal{S}_{\gamma_1}], \text{ where for some } j = 1, 2, \dots, l$$

$$\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_j} m_j^{(1)} \text{ and } \gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1,$$

$$\text{with } m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}, m_j = m_j^{(1)} + m_j^{(2)}.$$

and if $\beta = \omega^{\omega^\kappa}$ and γ is a limit ordinal with $\gamma \leq \beta$, then

$$(20) \quad \text{there is a sequence } (\eta(\gamma, n))_n \text{ increasing to } \gamma \text{ so that } \lambda(\beta\gamma, n) = \beta\eta(\gamma, n)$$

(this sequence $(\eta(\gamma, n))_n$ can depend on β).

Proof. We first will prove (18), (19) simultaneously for all $\gamma < \omega_1$. Assume that our claim is true all $\tilde{\gamma} < \gamma$. (19) follows from Proposition 2.4.

If $l = m_1 = 1$ we deduce (18) from the choice of $\lambda(\gamma, n)$, $n \in \mathbb{N}$, in that case (see (14), (15) and (16)). If $l \geq 2$ or $m_2 \geq 2$ we deduce from (13) and the induction hypothesis that

$$\begin{aligned} \mathcal{S}_{\lambda(\gamma, n)} &= \mathcal{S}_{\omega^{\xi_k} m_k + \dots + \omega^{\xi_2} m_2 + \omega^{\xi_1} m_1 + \lambda(\omega^{\xi_1}, n)} \\ &= \mathcal{S}_{\lambda(\omega^{\xi_1}, n)}[\mathcal{S}_{\omega^{\xi_k} m_k + \dots + \omega^{\xi_2} m_2 + \omega^{\xi_1} m_1}] \\ &\subset \mathcal{S}_{\lambda(\omega^{\xi_1}, n+1)}[\mathcal{S}_{\omega^{\xi_k} m_k + \dots + \omega^{\xi_2} m_2 + \omega^{\xi_1} m_1}] = \mathcal{S}_{\lambda(\gamma, n+1)}, \end{aligned}$$

which verifies (18) also in that case

To verify (20) let $\kappa < \omega_1$ with $\beta = \omega^{\omega^\kappa} \geq \gamma$. Recall that by (16) $\lambda(\omega^{\omega^\kappa + \xi_1}, n) = \omega^{\omega^\kappa} \theta(\xi_1, n)$. For each n , define $\eta(\gamma, n) = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} (m_1 - 1) + \theta(\xi_1, n)$. We will show that $(\eta(\gamma, n))_{n \in \mathbb{N}}$ has the desired property. Note that the Cantor Normal Form of $\beta\gamma$ is $\beta\gamma = \omega^{\omega^\kappa + \xi_l} m_l + \omega^{\omega^\kappa + \xi_{l-1}} m_{l-1} + \dots + \omega^{\omega^\kappa + \xi_1} m_1$. Hence, by (12):

$$\begin{aligned} \lambda(\beta\gamma, n) &= \omega^{\omega^\kappa + \xi_l} m_l + \omega^{\omega^\kappa + \xi_{l-1}} m_{l-1} + \dots + \omega^{\omega^\kappa + \xi_1} (m_1 - 1) + \lambda(\omega^{\omega^\kappa + \xi_1}, n) \\ &= \omega^{\omega^\kappa + \xi_l} m_l + \omega^{\omega^\kappa + \xi_{l-1}} m_{l-1} + \dots + \omega^{\omega^\kappa + \xi_1} (m_1 - 1) + \omega^{\omega^\kappa} \theta(\xi_1, n) \\ &= \omega^{\omega^\kappa} (\omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} (m_1 - 1) + \theta(\xi_1, n)) = \beta\eta(\gamma, n). \end{aligned}$$

□

Remark. The proof of Proposition 2.6, in particular the definition of $(\eta(\gamma, n))_n$, implies the following. Let ξ be a countable ordinal number and $\gamma \leq \beta = \omega^{\omega^\xi}$ be a limit ordinal number. If $\gamma = \omega^{\xi_1}$, then

$$(21) \quad \eta(\gamma, n) = \theta(\xi_1, n), \text{ for all } n \in \mathbb{N}.$$

Otherwise, if the Cantor normal form of γ is

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} m_1$$

and $\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_j} m_j^{(1)}$, $\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1$, with $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, $m_j = m_j^{(1)} + m_j^{(2)}$, then we have

$$(22) \quad \eta(\gamma, n) = \gamma_1 + \eta(\gamma_2, n), \text{ for all } n \in \mathbb{N}.$$

Corollary 2.7. *If $\alpha < \omega_1$ is a limit ordinal it follows that*

$$(23) \quad \mathcal{S}_\alpha = \{A \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\} : A \in \mathcal{S}_{\lambda(\alpha, \min(A))}\} \cup \{\emptyset\}.$$

Remark 2.8. If we had defined \mathcal{S}_α by (23), for limit ordinals $\alpha < \omega_1$ where $(\lambda(\alpha, n) : n \in \mathbb{N})$ is any sequence increasing to α , then we would not have ensured that the family \mathcal{S}_α is a regular family.

2.3. The fine Schreier families. We will now define the *Fine Schreier Sets*. For that we will also need to choose appropriate approximating sequences for limit ordinals. We will define them as a doubly indexed family $\mathcal{F}_{\beta,\alpha} \subset [\mathbb{N}]^{<\omega}$, $\alpha \leq \beta < \omega_1$. Later in the proof of Theorems A and C, we will fix β , depending on the Banach space X we are considering.

Definition 2.9. For a countable ordinal number ξ and $\beta = \omega^{\omega^\xi}$, we recursively define an hierarchy of families of finite subsets of the natural numbers $(\mathcal{F}_{\beta,\gamma})_{\gamma \leq \beta}$ as follows:

- (i) $\mathcal{F}_{\beta,0} = \{\emptyset\}$,
- (ii) if $\gamma < \beta$ then $\mathcal{F}_{\beta,\gamma+1} = \{\{n\} \cup F : F \in \mathcal{F}_{\beta,\gamma}, n \in \mathbb{N}\}$ (i.e., $\mathcal{F}_{\beta,\gamma+1} = \mathcal{F}_{\beta,1} \sqcup_{<} \mathcal{F}_{\beta,\gamma}$), and
- (iii) if $\gamma \leq \beta$ is a limit ordinal number, then $\mathcal{F}_{\beta,\gamma} = \bigcup_{n \in \mathbb{N}} (\mathcal{F}_{\beta,\eta(\gamma,n)} \cap [n, \infty)^{<\omega})$, where $(\eta(\gamma,n))_n$ is the sequence provided by Proposition 2.6 (and depends on β).

Remark. It can be easily shown by transfinite induction that each family $\mathcal{F}_{\beta,\gamma}$ is regular. In the literature fine Schreier families are usually defined recursively as a singly indexed family $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$ of subsets of $[\mathbb{N}]^{<\omega}$. In that case \mathcal{F}_0 and \mathcal{F}_α are defined for successor ordinals as in (i) and (ii). And if α is a limit ordinal is defined as in (iii) where we do not assume that the approximating sequence $(\eta(\alpha,n))_{n \in \mathbb{N}}$ does not depend of any $\beta \geq \alpha$.

Let ξ be a countable ordinal number and $\xi_1 \leq \omega^\xi$. If $\beta = \omega^{\omega^\xi}$ and $\gamma = \omega^{\xi_1}$, it follows by (21) that $\eta(\gamma,n) = \theta(\gamma,n)$ for $n \in \mathbb{N}$. The choice of $(\theta(\xi_1,n))_{n \in \mathbb{N}}$ may be done so that alongside (15), we also have

$$(24) \quad \mathcal{F}_{\beta,\eta(\gamma,n)} = \mathcal{F}_{\beta,\theta(\xi_1,n)} \subset \mathcal{F}_{\beta,\theta(\xi_1,n+1)} = \mathcal{F}_{\beta,\eta(\gamma,n+1)}.$$

This can be achieved by possibly adding to $\theta'(\xi_1,n)$ a large enough natural number.

Proposition 2.10. *Let ξ be a countable ordinal number and $\beta = \omega^{\omega^\xi}$. Assume that for all limit ordinals $\gamma \leq \beta$ the approximating sequence $(\eta(\gamma,n))_n$ satisfies conditions (21) and (22), and for the case $\gamma = \omega^{\xi_1}$ the approximating sequence $(\theta(\xi_1,n))_n$ satisfies condition (24). Then, for all $\gamma \leq \beta$, whose Cantor normal form is*

$$\gamma = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} m_1$$

we have that that

$$(25) \quad \mathcal{F}_{\beta,\eta(\gamma,n)} \subset \mathcal{F}_{\beta,\eta(\gamma,n+1)} \text{ for all } n \in \mathbb{N}, \text{ if } \gamma \text{ is a limit ordinal,}$$

and if for some $1 \leq j \leq l$, $\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_j} m_j^{(1)}$ and $\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1$ with $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, $m_j = m_j^{(1)} + m_j^{(2)}$, then

$$(26) \quad \mathcal{F}_{\beta,\gamma} = \mathcal{F}_{\beta,\gamma_2} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1}.$$

Proof. We will show (25) and (26) simultaneously by transfinite induction for all $\gamma \leq \beta$. Assume that (25) and (26) hold for all $\tilde{\gamma} < \gamma$. If $\gamma = \omega^\kappa$, then (25) follows from (21) and (24), while (26) is trivially satisfied. Indeed, in that case $\gamma = \gamma + 0 = 0 + \gamma$, are the only two choices to write γ as the sum of two ordinals, and we observe that $\mathcal{F}_{\beta,0} \sqcup_{<} \mathcal{F}_{\beta,\gamma} = \mathcal{F}_{\beta,\gamma} \sqcup_{<} \mathcal{F}_{\beta,0} = \mathcal{F}_{\beta,\gamma}$.

If γ is a limit ordinal (thus $\xi_1 > 0$) and either $l \geq 2$ or $m_l \geq 2$, then (25) follows from the inductive assumption. Indeed, for $n \in \mathbb{N}$ it follows that $\eta(\gamma,n) = \gamma' + \eta(\omega^{\xi_1},n)$ with $\gamma' = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} (m_1 - 1)$ and thus (note that $\gamma' + \eta(\omega^{\xi_1},n) < \gamma$)

$$\mathcal{F}_{\beta,\eta(\gamma,n)} = \mathcal{F}_{\beta,\eta(\omega^{\xi_1},n)} \sqcup_{<} \mathcal{F}_{\beta,\gamma'} \subset \mathcal{F}_{\beta,\eta(\omega^{\xi_1},n+1)} \sqcup_{<} \mathcal{F}_{\beta,\gamma'} = \mathcal{F}_{\beta,\eta(\gamma,n+1)}.$$

It is left to verify (26) in the case that $l \geq 2$ or $m_l \geq 2$. Let $\gamma = \gamma_1 + \gamma_2$ be a decomposition of γ as in the statement of (26). We can without loss of generality assume that $\gamma_2 > 0$.

If $\gamma_2 = \gamma'_2 + 1$ for some γ'_2 (which implies that γ itself is a successor ordinal) it follows from the inductive assumption and Definition 2.9 (ii) that

$$\begin{aligned} \mathcal{F}_{\beta,\gamma} &= \mathcal{F}_{\beta,\gamma_1+\gamma'_2+1} = \mathcal{F}_{\beta,1} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1+\gamma'_2} = \mathcal{F}_{\beta,1} \sqcup_{<} \left(\mathcal{F}_{\beta,\gamma'_2} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1} \right) \\ &= \left(\mathcal{F}_{\beta,1} \sqcup_{<} \mathcal{F}_{\beta,\gamma'_2} \right) \sqcup_{<} \mathcal{F}_{\beta,\gamma_1} = \mathcal{F}_{\beta,\gamma'_2+1} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1} = \mathcal{F}_{\beta,\gamma_2} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1}. \end{aligned}$$

If γ_2 is a limit ordinal then recall that by (22) we have $\eta(\gamma, n) = \gamma_1 + \eta(\gamma_2, n)$. If $A \in \mathcal{F}_{\beta,\gamma}$ it follows that there is an $n \leq \min A$ so that, using the inductive assumption, we have

$$A \in \mathcal{F}_{\beta,\eta(\gamma,n)} = \mathcal{F}_{\beta,\gamma_1+\eta(\gamma_2,n)} = \mathcal{F}_{\beta,\eta(\gamma_2,n)} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1}.$$

This means that $A = A_1 \cup A_2$ with, $A_1 < A_2$, $A_1 \in \mathcal{F}_{\beta,\eta(\gamma_2,n)}$, and $A_2 \in \mathcal{F}_{\beta,\gamma_1}$. If $A_1 \neq \emptyset$, then $\min(A_1) = \min(A) \geq n$, i.e. $A_1 \in \mathcal{F}_{\beta,\gamma_2}$ and hence $A \in \mathcal{F}_{\beta,\gamma_2} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1}$. If on the other hand $A_1 = \emptyset$, then $A \in \mathcal{F}_{\beta,\gamma_1} \subset \mathcal{F}_{\beta,\gamma_2} \sqcup_{<} \mathcal{F}_{\beta,\gamma_1}$.

Conversely, we can similarly show that if $A \in \mathcal{F}_{\beta,\gamma_1} \sqcup_{<} \mathcal{F}_{\beta,\gamma_2}$, then $A \in \mathcal{F}_{\beta,\gamma}$. \square

Corollary 2.11. *Let ξ be a countable ordinal number and $\gamma \leq \beta = \omega^{\omega^\xi}$ be a limit ordinal number. Then*

$$(27) \quad \mathcal{F}_{\beta,\gamma} = \{F \in [\mathbb{N}]^{<\omega} : F \in \mathcal{F}_{\beta,\eta(\gamma,\min(F))}\} \cup \{\emptyset\}.$$

The following formula of the Cantor Bendixson index of \mathcal{S}_α and $\mathcal{F}_{\beta,\alpha}$ can be easily shown by transfinite induction.

Proposition 2.12. *For any $\alpha, \kappa < \omega_1$, with $\alpha \leq \beta = \omega^{\omega^\kappa}$,*

$$\text{CB}(\mathcal{S}_\alpha) = \omega^\alpha + 1 \text{ and } \text{CB}(\mathcal{F}_{\beta,\alpha}) = \alpha + 1.$$

Moreover, assuming $\omega^\alpha \leq \beta$, for every $M \in [\mathbb{N}]^\omega$, there is an $N \in [M]^\omega$ so that

$$\mathcal{S}_\alpha^N \subset \mathcal{F}_{\beta,\omega^\alpha} \text{ and } \mathcal{F}_{\beta,\omega^\alpha}^N \subset \mathcal{S}_\alpha.$$

The main result in [15] states that if \mathcal{F} and \mathcal{G} are two hereditary subsets of $[\mathbb{N}]$, then for any $M \in [\mathbb{N}]^\omega$ there is an $N \in [M]^\omega$ so that $\mathcal{F} \cap [N]^{<\omega} \subset \mathcal{G}$ or $\mathcal{G} \cap [N]^{<\omega} \subset \mathcal{F}$. Together with Proposition 2.12 this yields

Proposition 2.13. *For any $\alpha, \gamma, \kappa < \omega_1$, with $\omega^\alpha \leq \beta = \omega^{\omega^\kappa}$, and any $M \in [N]^{<\omega}$, there is an $N \in [M]^{<\omega}$ so that*

$$\mathcal{S}_\gamma^N \subset \mathcal{S}_\alpha \cap [N]^{<\omega} \subset \mathcal{F}_{\beta,\omega^\alpha}, \text{ if } \gamma < \omega^\alpha, \text{ and } \mathcal{F}_{\beta,\omega^\alpha}^N \subset \mathcal{F}_{\beta,\omega^\alpha} \cap [N]^{<\omega} \subset \mathcal{S}_\gamma, \text{ if } \gamma > \omega^\alpha.$$

2.4. Families indexed by subsets of $[\mathbb{N}]^{<\omega}$. We consider families of the form $(x_A : A \in \mathcal{F})$ in some set X indexed over $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$. If \mathcal{F} is a tree, i.e., closed under restrictions, such a family is called an *indexed tree*. Let us also assume that \mathcal{F} is spreading. The passing to a *pruning* of such an indexed tree is what corresponds to passing to subsequences for sequences. Formally speaking we define a pruning of $(x_A : A \in \mathcal{F})$ as follows. Let $\pi : \mathcal{F} \rightarrow \mathcal{F}$ be an order isomorphism with the property that if $F \in \mathcal{F}$ is not maximal, then for any $n \in \mathbb{N}$, so that $n > \max(A)$ and $A \cup \{n\} \in \mathcal{F}$, $\pi(A \cup \{n\})$ is of the form $\pi(A) \cup \{s_n\}$, where (s_n) is a sequence which increases with n . We call then the family $(x_A : A \in \pi(\mathcal{F}))$ a *pruning* of $(x_A : A \in \mathcal{F})$. Let $\tilde{x}_A = x_{\pi(A)}$ for $A \in \mathcal{F}$. $(\tilde{x}_A : A \in \mathcal{F})$ is then simply a *relabeling* of the family $(x_A : A \in \pi(\mathcal{F}))$, and we call it also a pruning of $(x_A : A \in \mathcal{F})$. It is important to note that the branches of a pruning of an indexed tree $(x_A : A \in \mathcal{F})$, are a

subset of the branches of the original tree $(x_A : A \in \mathcal{F})$. Here a branch of $(x_A : A \in \mathcal{F})$, is a set of the form

$$\bar{x}_F = (x_{\{a_1\}}, x_{\{a_1, a_2\}}, \dots, x_{\{a_1, a_2, \dots, a_l\}}) \text{ for } F = \{a_1, a_2, \dots, a_l\} \in \mathcal{F}.$$

Also the nodes of a pruned tree, namely the sequences of the form $(\tilde{x}_{A \cup \{n\}} : A \cup \{n\} \in \mathcal{F})$, with $A \in \mathcal{F}$ not maximal, are subsequences of the nodes of the original tree.

Let us finally mention, how we usually choose prunings inductively. Let $\{A_n : n \in \mathbb{N}\}$ be a *consistent enumeration* of \mathcal{F} . By this we mean that if $\max(A_m) < \max(A_n)$ then $m < n$. Thus, we also have if $A_m \prec A_n$, then $m < n$, and if $A_m = A \cup \{s\} \in \mathcal{F}$ and $A_n = A \cup \{t\} \in \mathcal{F}$, for some (non maximal) $A \in \mathcal{F}$ and $s < t$ in \mathbb{N} , then $m < n$. Of course, then $A_1 = \emptyset$ and $\pi(\emptyset) = \emptyset$ assuming now that $\pi(A_1), \pi(A_2), \dots, \pi(A_m)$ have been chosen, then A_{m+1} must be of the form $A_m = A_l \subset \{k\}$, with $l < m = 1$. Moreover if, $k > \max(A_l) + 1$ and if $A_l \cup \{k-1\} \in \mathcal{F}$ then $A_l \cup \{k-1\} = A_j$ with $l < j < m+1$, and $\pi(A_j) = \pi(A_l) \cup \{s\}$ for some s has already been chosen. Thus, we need to choose $\pi(A_{m+1})$ to be of the form $\pi(A_l) \cup \{t\}$, where, in case that $A_l \cup \{k-1\} \in \mathcal{F}$, we need to choose $t > s$.

Proposition 2.14. *Assume that $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is compact. Let $r \in \mathbb{N}$ and $f : \text{MAX}(\mathcal{F}) \rightarrow \{1, 2, \dots, r\}$. Then for every $M \in [\mathbb{N}]^\omega$ there exists an $N \in [M]^\omega$ and an $i \in \{1, 2, \dots, r\}$ so that*

$$\text{MAX}(\mathcal{F}) \cap [N]^\omega \subset \{A \in \text{MAX}(\mathcal{F}) : f(A) = i\}.$$

Proposition 2.14 could be deduced from Corollary 2.5 and Proposition 2.6 in [3]. To make the paper as self-contained as possible we want to give a short proof.

Proof of 2.14. Without loss of generality we can assume that $r = 2$. We prove our assumption by induction on the Cantor Bendixson index of \mathcal{F} . If $\text{CB}(\mathcal{F}) = 1$, then \mathcal{F} can only consist of finitely many sets. Since \mathcal{F} is spreading it follows that $\mathcal{F} = \{\emptyset\}$ and our claim is trivial. Assume that our claim is true for all regular families \mathcal{E} , with $\text{CB}(\mathcal{E}) < \beta$. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where \mathcal{F} is a regular family with $\text{CB}(\mathcal{F}) = \alpha + 1$, where α is the predecessor of β , if β is a successor ordinal (and thus $\beta = \alpha + 1$), and $\alpha = \beta$, if β is a limit ordinal, and $f : \mathcal{F} \rightarrow \{1, 2\}$ with $\mathcal{F}_1 = \{A \in \mathcal{F} : f(A) = 1\}$ and $\mathcal{F}_2 = \{A \in \mathcal{F} : f(A) = 2\}$, and let $M \in [\mathbb{N}]^\omega$.

First we observe that there is a cofinite subset M' of M , so that $\text{CB}(\mathcal{F}(\{m\})) \leq \alpha$, for all $m \in M'$. Indeed, if that were not true, and thus $\text{CB}(\mathcal{F}(\{m\})) = \alpha + 1$ for all m in an infinite subsets M' of M , we could choose for every $m \in M'$ an element $A_m \in (\mathcal{F}(\{m\}))^{(\alpha)} = (\mathcal{F}^{(\alpha)})(\{m\})$. Thus $\{m\} \cup A_m \in \mathcal{F}^{(\alpha)}$. But this would imply that $\emptyset = \lim_{m \in M', m \rightarrow \infty} \{m\} \cup A_m \in \mathcal{F}^{(\alpha+1)}$, which is a contradiction.

Since $\text{CB}(\mathcal{F}(\{m\}))$ has to be a successor ordinal we deduce that $\text{CB}(\mathcal{F}(\{m\})) < \beta$, for $m \in M$, and can therefore apply our induction hypothesis, and choose inductively natural numbers $m_1 < m_2 < m_3 < \dots$ and infinite sets $N_0 = M' \supset N_1 \supset N_2 \dots$, and elements $c_1, c_2, \dots \in \{1, 2\}$, so that $m_j = \min N_{j-1} < \min(N_j)$ and $\mathcal{F}(\{m_j\}) \cap [N_j]^{<\omega} \subset \{A \in \mathcal{F} : f(A) = c_j\}$, for all $j \in \mathbb{N}$. Indeed, if N_{j-1} has been defined we let $m_j = \min(N_{j-1})$, and apply our induction hypothesis to the family $\mathcal{F}(\{m_j\})$, and the coloring $f_j : \text{MAX}(\mathcal{F}(\{m_j\})) \rightarrow \{1, 2\}$, $B \mapsto f(\{m_j\} \cup B)$, and the set $N_{j-1} \setminus \{m_j\}$, to obtain an infinite set $N_j \subset N_{j-1} \setminus \{m_j\}$.

Then take a c for which $\{j : c_j = c\}$ is infinite and $N = \{m_j : c_j = c\}$. If $A = \{a_1, a_2, \dots, a_l\} \in \text{MAX}(\mathcal{F}) \cap [N]^{<\omega}$, Then $a_1 = m_j$ for some $j \in \mathbb{N}$, with $c_j = c$, and $\{a_2, a_3, \dots, a_n\} \in \text{MAX}(\mathcal{F}(\{a_1\})) \cap [N_j]^{<\omega}$, and thus $f(A) = c$, which verifies our claim. \square

3. REPEATED AVERAGES ON SCHREIER SETS

We recall *repeated averages* defined on maximal sets of \mathcal{S}_α , $\alpha < \omega_1$ (c.f. [4]). As in our previous sections we will assume that \mathcal{S}_α is recursively defined using the conditions made in Subsection 2.2. We first need the following characterization of maximal elements of \mathcal{S}_α , $\alpha < \omega_1$, which can be easily proven by transfinite induction using for the limit ordinal case Corollary 2.7.

Proposition 3.1. *Let $\alpha < \omega_1$ then*

- (i) *$A \in \text{MAX}(\mathcal{S}_{\alpha+1})$ if and only if $A = \bigcup_{j=1}^n A_j$, with $n = \min(A_1)$ and $A_1 < A_2 < \dots < A_n$ are in $\text{MAX}(\mathcal{S}_\alpha)$. In this case the A_j are unique.*
- (ii) *If α is a limit ordinal then $A \in \text{MAX}(\mathcal{S}_\alpha)$ if and only if $A \in \text{MAX}(\mathcal{S}_{\lambda(\alpha, \min(A))})$.*

For each $\alpha < \omega_1$ and each $A \in \text{MAX}(\mathcal{S}_\alpha)$ we will now introduce an element $z_{(\alpha, A)} \in S_{\ell_1^+}$ with $\text{supp}(z_{(\alpha, A)}) = A$, which we will call *repeated average of complexity α on $A \in \text{MAX}(\mathcal{S}_\alpha)$* . If $\alpha = 0$ then $\text{MAX}(\mathcal{S}_0)$ consists of singletons and for $A = \{n\} \in \text{MAX}(\mathcal{S}_\alpha)$ we put $z_{(0, \{n\})} = e_n$, the n -th element of the unit basis of ℓ_1 . Assume for all $\gamma < \alpha$ and all $A \in \text{MAX}(\mathcal{S}_\gamma)$ we already introduced $z_{(\gamma, A)}$ which we write as $z_{(\gamma, A)} = \sum_{a \in A} z_{(\gamma, A)}(a)e_a$, with $z_{(\gamma, A)} > 0$ for all $a \in A$. If $\alpha = \gamma + 1$ for some $\gamma < \omega_1$ and if $A \in \text{MAX}(\mathcal{S}_\alpha)$ we write by Proposition 3.1 (i) A in a unique way as $A = \bigcup_{j=1}^n A_j$, with $n = \min A$ and $A_1 < A_2 < \dots < A_n$ are maximal in \mathcal{S}_γ . We then define

$$(28) \quad z_{(\alpha, A)} = \frac{1}{n} \sum_{j=1}^n z_{(\gamma, A_j)} = \frac{1}{n} \sum_{j=1}^n \sum_{a \in A_j} z_{(\gamma, A_j)}(a)e_a,$$

and thus

$$(29) \quad z_{(\alpha, A)}(a) = \frac{1}{n} z_{(\gamma, A_j)}(a) \text{ for } j = 1, 2, \dots, n \text{ and } a \in A_j.$$

If α is a limit ordinal and $A \in \text{MAX}(\mathcal{S}_\alpha)$ then, by Corollary 2.7, $A \in \mathcal{S}_{\lambda(\alpha, \min(A))}$, and we put

$$(30) \quad z_{(\alpha, A)} = z_{(\lambda(\alpha, \min(A)), A)} = \sum_{a \in A} z_{(\lambda(\alpha, \min(A)), A)}(a)e_a.$$

The following result was, with slightly different notation, proved in [4].

Lemma 3.2. [4, Proposition 2.15] *For all $\varepsilon > 0$, all $\gamma < \alpha$, and all $M \in [\mathbb{N}]^\omega$, there is an $N = N(\gamma, \alpha, M, \varepsilon) \in [M]^\omega$, so that $\sum_{a \in A} z_{(\alpha, B)}(a) < \varepsilon$ for all $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^\omega)$ and $A \in \mathcal{S}_\gamma$.*

The following Proposition will be proved by transfinite induction.

Proposition 3.3. *Assume $\alpha < \omega_1$ and $A \in \mathcal{S}_\alpha$ (not necessarily maximal). If B_1, B_2 are two extensions of A which both are maximal in \mathcal{S}_α then it follows*

$$z_{(\alpha, B_1)}(a) = z_{(\alpha, B_2)}(a) \text{ for all } a \in A.$$

Remark. Proposition 3.3 says the following: If $\alpha < \omega_1$ and $A = \{a_1, a_2, \dots, a_n\}$ is in $\text{MAX}(\mathcal{S}_\alpha)$, then $z_{(\alpha, A)}(a_1)$ only depends on a_1 , $z_{(\alpha, A)}(a_2)$, only depends on a_1 and a_2 etc.

Proof of Proposition 3.3. Our claim is trivial for $\alpha = 0$, assume that $\alpha = \gamma + 1$ and our claim is true for γ , and let $A \in \mathcal{S}_{\gamma+1}$. Without loss of generality $A \neq \emptyset$, otherwise we would be done. Using Proposition 3.1, we can find an integer $1 \leq l \leq \min A$, sets $A_1, A_2, \dots, A_{l-1} \in$

$\text{MAX}(S_\gamma)$, and $A_l \in S_\gamma$ (not necessarily maximal in S_γ) so that $A_1 < A_2 < \dots < A_l$ and $A = \bigcup_{j=1}^l A_j$. By Proposition 3.1, any extension of A to a maximal element in S_γ will then be of the form $B = \bigcup_{j=1}^l A_j \cup \bigcup_{j=l}^{\min(A)} B_j$, where $A_l < B_l < B_{l+1} < \dots < B_{\min(A)}$ (B_l may be empty, in which case $A_l < B_{l+1} < \dots < B_{\min(A)}$), so that $A_l \cup B_l, B_{l+1}, \dots, B_{\min(A)}$ are in $\text{MAX}(S_\gamma)$. No matter how we extend A to a maximal element B in $S_{\gamma+1}$, the inductive formula (28) yields

$$z_{(\gamma+1,B)}(a) = \frac{1}{\min(A)} z_{(\gamma,A_j)}(a) \text{ whenever for some } j = 1, 2, \dots, l-1 \text{ we have } a \in A_j.$$

In the case that $a \in A_l$, then, by our induction hypothesis, $z_{\gamma,A_l \cup B_l}(a)$ does not depend on the choice of B_l , and

$$z_{(\gamma+1,B)}(a) = \frac{1}{n} z_{(\gamma,A_l \cup B_l)}(a) \text{ whenever } a \in A_l$$

Thus, in both cases, the value of $z_{(\gamma+1,B)}(a)$ does not depend on how we extend A to a maximal element B in $S_{\gamma+1}$.

If α is a limit ordinal and $A \in S_\alpha$ is not maximal, we also can assume that $A \neq \emptyset$, and thus it follows from the formula (23) in Corollary (2.7) that $A \in S_{\lambda(\alpha, \min(A))}$. For any two extension B of A into a maximal set of $\text{MAX}(S_\alpha)$, it follows from Proposition 3.1 that B is maximal in $S_{\lambda(\alpha, \min(A))}$, and that $z_{(\alpha,B)} = z_{(\lambda(\alpha, \min(A)), B)}$. Thus, also in this case our claim follows from the induction hypothesis. \square

Using Proposition 3.3 we can define consistently $z_{(\alpha,A)} \in B_{\ell_1^+}$ for any $\alpha < \omega_1$ and any $A \in S_\alpha$ by

$$z_{(\alpha,A)} = \sum_{a \in A} z_{(\alpha,B)}(a) e_a, \text{ where } B \text{ is any maximal extension of } A \text{ in } \text{MAX}(S_\alpha).$$

In particular this implies the following recursive definition of $z_{(\alpha,A)}$: If $A \in S_{\alpha+1} \setminus \{\emptyset\}$ we can write in a unique way A as $A = \bigcup_{j=1}^n A_j$, where $A_1 < A_2 < \dots < A_n$, $A_j \in \text{MAX}(S_\alpha)$, for $j = 1, 2, \dots, n-1$ and $A_n \in S_\alpha \setminus \{\emptyset\}$, and note that

$$(31) \quad z_{(\alpha+1,A)} = \frac{1}{\min(A)} \sum_{j=1}^n z_{(\alpha,A_j)}$$

and if α is a limit ordinal

$$(32) \quad z_{(\alpha,A)} = z_{(\alpha, \lambda(\alpha, \min(A)))}.$$

For $D \in S_\alpha$ define $\zeta(\alpha, A) = z_{(\alpha,D)}(\max(D))$. For $A \in S_\alpha$ it follows therefore

$$z_{(\alpha,D)} = \sum_{D \preceq A} \zeta(\alpha, D) e_{\max(D)}.$$

We also put $\zeta(\alpha, \emptyset) = 0$ and $e_{\max(\emptyset)} = 0$.

By transfinite induction we can easily show the following estimate for $1 \leq \alpha < \omega_1$

$$(33) \quad \zeta(\alpha, A) \leq \frac{1}{\min A}.$$

From Proposition 2.6 we deduce the following formula for $z_{(\alpha,A)}$

Proposition 3.4. *Assume $\alpha < \omega_1$ and that its Cantor Normal Form is*

$$\alpha = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_1} m_1.$$

Let $j = 1, 2, \dots, l$ and $m_j^{(1)}, m_j^{(2)} \in \mathbb{N} \cup \{0\}$, with $m_j^{(1)} + m_j^{(2)} = m_j$. Put

$$\gamma_1 = \omega^{\xi_l} m_l + \omega^{\xi_{l-1}} m_{l-1} + \dots + \omega^{\xi_{j-1}} m_{j-1} + \omega^{\xi_j} m_j^{(1)}$$

$$\gamma_2 = \omega^{\xi_j} m_j^{(2)} + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1.$$

For $A \in \text{MAX}(\mathcal{S}_\alpha)$ we use Proposition 2.6 and write $A = \bigcup_{j=1}^n A_j$, where $A_j \in \mathcal{S}_{\gamma_1}$ for $j = 1, 2, \dots, n$, $A_1 < A_2 < \dots < A_n$, and $B = \{\min(A_j) : j = 1, 2, \dots, n\} \in \mathcal{S}_{\gamma_2}$.

Then it follows that $A_j \in \text{MAX}(\mathcal{S}_{\gamma_1})$, for $j = 1, 2, \dots, n$, $B \in \text{MAX}(\mathcal{S}_{\gamma_2})$ and

$$(34) \quad z_{(\alpha, A)} = \sum_{j=1}^n z_{(\gamma_2, B)}(\min(A_j)) z_{(\gamma_1, A_j)}.$$

In other words, if $\emptyset \prec D \preceq A$, and thus $D = \bigcup_{j=1}^{i-1} A_j \cup \tilde{A}_i$, for some $0 \leq i < n$, and $\emptyset \prec \tilde{A}_i \preceq A_i$, then

$$(35) \quad \zeta(\alpha, D) = \zeta(\gamma_2, \{\min(A_j) : j = 1, 2, \dots, i\}) \cdot \zeta(\gamma_1, \tilde{A}_i).$$

Proof. We prove by transfinite induction for all $\beta < \omega_1$, with Cantor Normal Form

$$\beta = \omega^{\xi_j} m_j + \omega^{\xi_{j-1}} m_{j-1} + \dots + \omega^{\xi_1} m_1,$$

the following

Claim: If $\gamma < \omega_1$ has Cantor Normal Form

$$\gamma = \omega^{\xi_l} \tilde{m}_l + \omega^{\xi_{l-1}} \tilde{m}_{l-1} + \dots + \omega^{\xi_j} \tilde{m}_j,$$

where \tilde{m}_j could possibly be vanishing, and if $A = \bigcup_{i=1}^n A_i \in \text{MAX}(\mathcal{S}_{\gamma+\beta}) = \text{MAX}(\mathcal{S}_\beta[\mathcal{S}_\gamma])$, where $A_i \in \mathcal{S}_\gamma$ for $i = 1, 2, \dots, n$, $A_1 < A_2 < \dots < A_n$, and $B = \{\min(A_i) : i = 1, 2, \dots, n\} \in \mathcal{S}_\beta$, then it follows that $A_i \in \text{MAX}(\mathcal{S}_\gamma)$ for $i = 1, 2, \dots, n$, $B \in \text{MAX}(\mathcal{S}_{\gamma_1})$ and

$$(36) \quad z_{(\alpha, A)} = \sum_{i=1}^n z_{(\beta, B)}(\min(A_i)) z_{(\gamma, A_i)}$$

For $\beta = 0$ the claim is trivial and for $\beta = 1$, our claim follows from Proposition 3.1 and the definition of $z_{(\gamma+1, A)}$ for $A \in \text{MAX}(\mathcal{S}_{\gamma+1})$.

Assume now that the claim is true for all $\tilde{\beta} < \beta$, and that $\gamma < \omega_1$ has above form and let $A = \bigcup_{i=1}^n A_i \in \text{MAX}(\mathcal{S}_{\gamma+\beta})$, where $A_i \in \mathcal{S}_\gamma$ for $i = 1, 2, \dots, n$, $A_1 < A_2 < \dots < A_n$, and $B = \{\min(A_i) : i = 1, 2, \dots, n\} \in \mathcal{S}_\beta$.

First we note that (10) implies that the A_i are maximal in \mathcal{S}_γ . Indeed, if for some $i_0 = 1, 2, \dots, n$, A_{i_0} is not maximal in \mathcal{S}_γ , then if $i_0 = n$, it would directly follow that A cannot be maximal in $\mathcal{S}_{\gamma+\beta}$ and if $i_0 < n$ we could define $\tilde{A}_i = A_i$, for $i = 1, 2, \dots, l-1$, $\tilde{A}_{i_0} = A_{i_0} \cup \{\min(A_{i_0+1})\}$, $\tilde{A}_i = (A_i \cup \{\min(A_{i+1})\}) \setminus \{\min(A_i)\}$, for $i = i_0, i_0+1, \dots, l-1$, and $\tilde{A}_l = A_l \setminus \{\min A_l\}$. Then, by (10) and the fact that the Schreier families are spreading, $A = \bigcup_{i=1}^n \tilde{A}_i$ is also a decomposition of elements of \mathcal{S}_γ with $\tilde{B} = \{\min(\tilde{A}_i) : i = 1, 2, \dots, n\} \in \mathcal{S}_\beta$. But now \tilde{A}_n is not maximal in \mathcal{S}_γ and we get again a contradiction.

It is also easy to see that B is maximal in \mathcal{S}_β .

To verify (36) we first assume that β is a successor ordinal, say $\beta = \alpha + 1$. Then we can write B as $B = \bigcup_{i=1}^m B_i$, where $m = \min(B) = \min(A)$, $B_1 < B_2 < \dots < B_m$, and $B_i \in \text{MAX}(\mathcal{S}_\alpha)$, for $i = 1, 2, \dots, m$. We can write B_i as

$$B_i = \{\min(A_s) : s = k_{i-1} + 1, k_{i-1} + 2, \dots, k_i\},$$

with $k_0 = 0 < k_1 < \dots < k_m = n$. We put $\bar{A}_i = \bigcup_{s=k_{i-1}+1}^{k_i} A_s \in \mathcal{S}_{\gamma+\alpha} = \mathcal{S}_\alpha[\mathcal{S}_\gamma]$, for $i = 1, 2, \dots, m$. From the definition of $z_{(\beta+1, B)}$ and from the induction hypothesis we deduce now that

$$\begin{aligned} z_{(\gamma+\alpha+1, A)} &= \frac{1}{m} \sum_{i=1}^m z_{(\gamma+\alpha, \bar{A}_i)} \\ &= \frac{1}{m} \sum_{i=1}^m \sum_{s=k_{i-1}+1}^{k_i} z_{(\alpha, B_i)}(\min(A_s)) z_{(\gamma, A_s)} = \sum_{s=1}^n z_{(\beta, B)}(\min(A_s)) z_{(\gamma, A_s)}, \end{aligned}$$

which proves the claim if β is a successor ordinal.

If β is a limit ordinal it follows from Corollary 2.7, our definition of $z_{(\beta, B)}$ and $z_{(\gamma+\beta, A)}$, and our choice of the approximating sequence $(\lambda(\gamma + \beta), n)$ that

$$\begin{aligned} z_{(\gamma+\beta, A)} &= z_{(\lambda(\gamma+\beta, \min(A)), A)} = z_{(\gamma+\lambda(\beta, \min(B)), A)} \\ &= \sum_{j=1}^n z_{(\lambda(\beta, \min(B)), B)}(\min(A_j)) z_{(\gamma, A_j)} = \sum_{j=1}^n z_{(\beta, B)}(\min(A_j)) z_{(\gamma, A_j)} \end{aligned}$$

which proves our claim also in the limit ordinal case. \square

If $\alpha < \omega_1$ and $A \in \text{MAX}(\mathcal{S}_\alpha)$ then $z_{(\alpha, A)}$ is an element of $S_{\ell_1} \cap \ell_1^+$ and can therefore be seen as probability on \mathbb{N} , as well as A . We denote the expected value of a function f defined on A or on all of \mathbb{N} as $\mathbb{E}_{(\alpha, A)}(f)$.

As done in [33], we deduce the following statement from Lemma 3.2.

Corollary 3.5. [33, Corollary 4.10] *For each $\alpha < \omega_1$ and $A \in \text{MAX}(\mathcal{S}_\alpha)$ let $f_A : A \rightarrow [-1, 1]$ have the property that $\mathbb{E}_{\alpha, A}(f_A) \geq \rho$, for some fixed number $\rho \in [-1, 1]$. For $\delta > 0$ and $M \in [\mathbb{N}]^\omega$ put*

$$\mathcal{A}_{\delta, M} = \left\{ A \in \mathcal{S}_\alpha \cap [M]^{<\omega} : \begin{array}{l} \exists B \in \text{MAX}(\mathcal{S}_\alpha \cap [M]^{<\omega}), \\ A \subset B, \text{ and } f_B(a) \geq \rho - \delta \text{ for all } a \in A \end{array} \right\}.$$

Then $\text{CB}(\mathcal{A}_{\delta, M}) = \omega^\alpha + 1$.

We finish this section with an observation, which will be needed later.

Definition 3.6. If $A \subset \mathbb{N} \setminus \{\emptyset\}$ is finite, we can write it in a unique way as a union $A = \bigcup_{j=1}^d A_j$, where $A_1 < A_2 < \dots < A_d$, and $A_j \in \text{MAX}(\mathcal{S}_1)$ if $j = 1, 2, \dots, d-1$, and $A_d \in \mathcal{S}_1 \setminus \{\emptyset\}$. We call $(A_j)_{j=1}^d$ the *optimal \mathcal{S}_1 -decomposition of A* , and we define

$$l_1(A) = \min(A_d) - \#A_d.$$

For $A = \emptyset$ we put $l_1(A) = 0$.

The significance of this number and its connection to the repeated averages is explained in the following Lemma.

Lemma 3.7. *Let $\alpha \in [1, \omega_1)$, $A \in \mathcal{S}_\alpha$ and let $(A_j)_{j=1}^d$ be its optimal \mathcal{S}_1 -decomposition.*

- (i) $l_1(A) = 0$ if and only if $A = \emptyset$ or $A_d \in \text{MAX}(\mathcal{S}_1)$.

- (ii) If $A \in \text{MAX}(\mathcal{S}_\alpha)$ then $A_d \in \text{MAX}(\mathcal{S}_1)$ and, thus, $l_1(A) = 0$.
- (iii) If $l_1(A) > 0$, then for all $\max(A) < k_1 < k_2 < \dots < k_{l_1(A)}$ it follows that $A \cup \{k_1, k_2, \dots, k_{l_1(A)}\} \in \mathcal{S}_\alpha$ and

$$\zeta(\alpha, A \cup \{k_1, k_2, \dots, k_i\}) = \zeta(\alpha, A) \text{ for all } i = 1, 2 \dots l_1(A).$$

- (iv) If $m > l_1(A)$ and $\max(A) < k_1 < k_2 < \dots < k_m$, have the property that $A \cup \{k_1, k_2, \dots, k_m\} \in \mathcal{S}_\alpha$ then

$$\zeta(\alpha, A \cup \{k_1, k_2, \dots, k_i\}) \leq \frac{1}{k_{l_1(A)+1}}.$$

- (v) If $A \neq \emptyset$, then

$$\sum_{D \preceq A, l_1(D')=0} \zeta(\alpha, D) \leq \frac{1}{\min(A)} \text{ and } \sum_{D \preceq A, l_1(D)=0} \zeta(\alpha, D) \leq \frac{1}{\min(A)}$$

(recall that $D' = D \setminus \{\max D\}$ for $D \in [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$ and $\emptyset' = \emptyset$).

Proof. We proof (i) through (v) by transfinite induction for all $\alpha \in [1, \omega_1)$. For $\alpha = 1$, (i), (ii), (iii) and (v) follow from the definition of \mathcal{S}_1 and the definition of $\zeta(\alpha, A)$, for $A \in \mathcal{S}_1$, while (iv) is vacuous in that case. Assume our claim is true for some $\alpha < \omega_1$, and let $A \in \mathcal{S}_{\alpha+1}$. Without loss of generality we can assume that $A \neq \emptyset$. Indeed, if $A = \emptyset$, then (i) is clear (ii), (iii), and (v) are vacuous, while (iv) follows easily by induction from the fact that always $\zeta(\alpha, A) \leq \frac{1}{\min(A)}$ if $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$. By the definition of $\mathcal{S}_{\alpha+1}$, A can be written in a unique way as $A = \bigcup_{j=1}^n B_j$ where $B_j \in \text{MAX}(\mathcal{S}_\alpha)$, for $j = 1, 2, \dots, n-1$, and $B_n \in \mathcal{S}_\alpha$. For $j = 1, 2, \dots, n$ let $(A_{j,i})_{i=1}^{c_j}$ be the optimal \mathcal{S}_1 -decomposition of B_j . From the induction hypothesis (ii) it follows that $(A_{j,i})$ are maximal in \mathcal{S}_1 , for $j < n$ or for $j = n$ and $i < c_n$. Therefore it follows that $(A_{j,i} : j = 1, 2, \dots, n, i = 1, 2 \dots c_j)$ (appropriately ordered) is the optimal \mathcal{S}_1 -decomposition of A and it follows $l_1(A) = l_1(B_n)$, and $A_d = A_{n,c_n}$.

We can deduce (i) from the induction hypothesis. If $A \in \text{MAX}(\mathcal{S}_{\alpha+1})$, then, in particular, $B_n \in \text{MAX}(\mathcal{S}_\alpha)$, and thus $l_1(A) = l_1(B_n) = 0$. Conversely, if $l_1(A) = l_1(B_n) = 0$, then $A_d = A_{n,c_n} \in \text{MAX}(\mathcal{S}_1)$. This proves (ii) for $\alpha + 1$.

If $l_1(A) > 0$ and $\max(A) = \max(B_n) < k_1 < k_2 < \dots < k_{l_1(A)}$, then it follows from the fact that $l_1(A) = l_1(B_n)$ and our induction hypothesis that $B_n \cup \{k_1, k_2, \dots, k_{l_1(A)}\} \in \mathcal{S}_\alpha$ and

$$\zeta(\alpha, B_n \cup \{k_1, k_2, \dots, k_i\}) = \zeta(\alpha, B_n).$$

Therefore $A \cup \{k_1, k_2, \dots, k_{l_1(A)}\} \in \mathcal{S}_{\alpha+1}$ and, using our recursive formula, we obtain

$$\zeta(\alpha, A \cup \{k_1, k_2, \dots, k_i\}) = \frac{1}{\min(A)} \zeta(\alpha, B_n \cup \{k_1, k_2, \dots, k_i\}) = \frac{1}{\min(A)} \zeta(\alpha, B_n) = \zeta(\alpha, A),$$

which verifies (iii). In order to show (iv) let $m > l_1(A)$ and $\max(A) < k_1 < k_2 < \dots < k_m$, is such that $A \cup \{k_1, k_2, \dots, k_m\} \in \mathcal{S}_{\alpha+1}$ we distinguish between two cases. Either $B_n \cup \{k_1, k_2, \dots, k_m\} \in \mathcal{S}_\alpha$. In that case we deduce from the induction hypothesis

$$\zeta(\alpha, A \cup \{k_1, k_2, \dots, k_m\}) = \frac{1}{\min(A)} \zeta(\alpha, B_n \cup \{k_1, k_2, \dots, k_m\}) \leq \frac{1}{k_{l_1(A)+1}}.$$

Or we can write $A \cup \{k_1, k_2, \dots, k_m\}$ as $A \cup \{k_1, k_2, \dots, k_m\} = \bigcup_{j=1}^n B_j \cup \bigcup_{j=n}^p B'_j$, where $p > n$, $B_n < B'_n < B'_{n+1} < \dots < B'_p$, $B_n \cup B'_n \in \text{MAX}(\mathcal{S}_\alpha)$, $B'_{n+1}, \dots, B'_{p-1} \in \text{MAX}(\mathcal{S}_\alpha)$,

and $B'_p \in \mathcal{S}_\alpha \setminus \{\emptyset\}$. Let $s \leq m$ such that $k_s = \min(B_p)$. Then $s > l_1(B_n)$, and $l_1(B_n \cup B'_n) = 0$. It follows therefore from (31) and the induction hypothesis that

$$\zeta(\alpha + 1, A \cup \{k_1, k_2, \dots, k_m\}) = \frac{1}{\min(A)} \zeta(\alpha, \{k_s, k_{s+1}, \dots, k_m\}) \leq \frac{1}{k_s} \leq \frac{1}{k_{l_1(A)+1}},$$

This proves (iv) in both cases.

Finally, to verify (v) we observe, that by the induction hypothesis and (31)

$$\begin{aligned} \sum_{D \preceq A, l_1(D')=0} \zeta(\alpha + 1, D) &= \frac{1}{\min(A)} \sum_{j=1}^n \sum_{D \preceq B_j, l_1(D')=0} \zeta(\alpha, D) \\ &\leq \frac{1}{\min A} \sum_{j=1}^n \frac{1}{\min(B_j)} \leq \frac{n}{\min(A)} \frac{1}{\min(A)} \leq \frac{1}{\min(A)}, \end{aligned}$$

which proves the first part of (v), while the second follows in the same way.

If $\alpha < \omega_1$ is a limit ordinal and assuming that our claim is true for all $\gamma < \alpha$ we proceed as follows. For $A \in \mathcal{S}_\alpha$, we can assume again that $A \neq \emptyset$ and it follows from Corollary (2.7) that $A \in \mathcal{S}_{\lambda(\alpha, \min(A))}$ and, by Proposition 3.1 A is maximal in \mathcal{S}_α if and only if it is maximal in $\mathcal{S}_{\lambda(\alpha, \min(A))}$. Therefore (i) through (v) follow from our claim being true for $\lambda(\alpha, \min(A))$. \square

Remark. Recall, that if $\beta = \omega^\omega$ is a countable ordinal number and $\gamma < \beta$, then by (17) we have $\mathcal{S}_{\beta(\gamma+1)} = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$. An argument very similar to what was used in the proof of Lemma 3.7, implies the following: if $B_1 < \dots < B_d$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ so that $\bar{B} = \{\min(B_j) : 1 \leq j \leq d\}$ is a non-maximal \mathcal{S}_β set, $D = \cup_{j=1}^d B_j$ and $C \in \mathcal{S}_{\beta\gamma}$ with $D < C$, then

$$(37) \quad l_1(C) = l_1(D \cup C).$$

Corollary 3.8. *Let $A = \{a_1, a_2, \dots, a_l\}$ and $\tilde{A} = \{\tilde{a}_1, \dots, \tilde{a}_{\tilde{l}}\}$ be two sets in $[\mathbb{N}]^{<\omega}$ whose optimal \mathcal{S}_1 -decompositions $(A_j)_{j=1}^d$ and $(\tilde{A}_j)_{j=1}^{\tilde{d}}$, respectively, have the same length and satisfy $\min(A_j) = \min(\tilde{A}_j)$, for $j = 1, 2, \dots, d$.*

Then it follows for $\alpha < \omega_1$ that $A \in \mathcal{S}_\alpha$ if and only if $\tilde{A} \in \mathcal{S}_\alpha$ and in that case for $D \preceq A$ and $\tilde{D} \preceq \tilde{A}$, with $\#D = \#\tilde{D}$, it follows that $\zeta(\alpha, D) = \zeta(\alpha, \tilde{D})$.

Proof. We prove this lemma by transfinite induction on α . If $\alpha = 1$ then, $A_1 = A$ and $\tilde{A}_1 = \tilde{A}$, and $a_1 = \tilde{a}_1$, and thus $\zeta(1, D) = \zeta(1, \tilde{D})$ for all $D \preceq A$ and $\tilde{D} \prec \tilde{A}$.

Assume that the conclusion holds for some α and let $A \in \mathcal{S}_{\alpha+1}$, $\tilde{A} \in [\mathbb{N}]^{<\omega}$ satisfy the assumption. Let $A = \cup_{i=1}^p C_i$, where $C_1 < \dots < C_{p-1}$ are in $\text{MAX}(\mathcal{S}_\alpha)$, whereas $C_p \in \mathcal{S}_\alpha$ and $p \leq \min(A)$. Write also \tilde{A} as $\tilde{A} = \cup_{i=1}^p \tilde{C}_i$, where $\tilde{C}_1 < \dots < \tilde{C}_p$, and the \tilde{C}_j are chosen such that $\#C_j = \#\tilde{C}_j$ for $j = 1, 2, \dots, p-1$.

From Lemma 3.7 (ii) it follows that for some sequence $0 = d_0 < d_1 < d_2 < \dots < d_p = d$ the sequence $(A_j)_{j=d_{i-1}+1}^{d_i}$ is the optimal \mathcal{S}_1 -decomposition of C_i for $i = 1, 2, \dots, p$. Now we can first deduce $(\tilde{A}_j)_{j=1}^{d_1}$ is the optimal \mathcal{S}_1 -decomposition of \tilde{C}_1 , then deduce that $(\tilde{A}_j)_{j=d_1+1}^{d_2}$ is the optimal \mathcal{S}_1 -decomposition of \tilde{C}_2 , and so on. We are therefore in the position to apply the induction hypothesis and deduce that for all $i = 1, 2, \dots, p$, and $D \preceq C_i$ and $\tilde{D} \preceq \tilde{C}_i$ it follows that $\zeta(\alpha, D) = \zeta(\alpha, \tilde{D})$. Our claim follows therefore from our recursive formula (31).

As usual in the case that α is a limit ordinal the verification follows easily from the definition of \mathcal{S}_α in that case. \square

Lemma 3.9. *Let X be a Banach space, α be a countable ordinal number, $B \in \text{MAX}(\mathcal{S}_\alpha)$ and $(x_A)_{A \preceq B}$ be vectors in B_X . Then*

$$(38) \quad \left\| \sum_{A \preceq B} \zeta(\alpha, A)x_A - \sum_{A \preceq B} \zeta(\alpha, A')x_A \right\| \leq \frac{2}{\min(B)}.$$

Proof. Using, Lemma 3.7 part (iii), then part (v) we obtain

$$\begin{aligned} & \left\| \sum_{A \preceq B} \zeta(\alpha, A)x_A - \sum_{A \preceq B} \zeta(\alpha, A')x_A \right\| \\ & \leq \left\| \sum_{\substack{A \preceq B \\ l_1(A') \neq 0}} (\zeta(\alpha, A) - \zeta(\alpha, A'))x_A \right\| + \left\| \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A)x_A \right\| + \left\| \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A')x_A \right\| \\ & \leq \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A) + \sum_{\substack{A \preceq B \\ l_1(A') = 0}} \zeta(\alpha, A') \leq \frac{2}{\min(B)}. \end{aligned}$$

□

4. TREES AND THEIR INDICES

Let X be an arbitrary set. We set $X^{<\omega} = \bigcup_{n=0}^{\infty} X^n$, the set of all finite sequences in X , which includes the sequence of length zero denoted by \emptyset . For $x \in X$ we shall write x instead of (x) , *i.e.*, we identify X with sequences of length 1 in X . A *tree on X* is a non-empty subset \mathcal{F} of $X^{<\omega}$ closed under taking initial segments: if $(x_1, \dots, x_n) \in \mathcal{F}$ and $0 \leq m \leq n$, then $(x_1, \dots, x_m) \in \mathcal{F}$. A tree \mathcal{F} on X is *hereditary* if every subsequence of every member of \mathcal{F} is also in \mathcal{F} .

Given $\bar{x} = (x_1, \dots, x_m)$ and $\bar{y} = (y_1, \dots, y_n)$ in $X^{<\omega}$, we write (\bar{x}, \bar{y}) for the concatenation of \bar{x} and \bar{y} :

$$(\bar{x}, \bar{y}) = (x_1, \dots, x_m, y_1, \dots, y_n).$$

Given $\mathcal{F} \subset X^{<\omega}$ and $\bar{x} \in X^{<\omega}$, we let

$$\mathcal{F}(\bar{x}) = \{\bar{y} \in X^{<\omega} : (\bar{x}, \bar{y}) \in \mathcal{F}\}.$$

Note that if \mathcal{F} is a tree on X , then so is $\mathcal{F}(\bar{x})$ (unless it is empty). Moreover, if \mathcal{F} is hereditary, then so is $\mathcal{F}(\bar{x})$ and $\mathcal{F}(\bar{x}) \subset \mathcal{F}$.

Let X^ω denote the set of all (infinite) sequences in X . Fix $S \subset X^\omega$. For a subset \mathcal{F} of $X^{<\omega}$ the *S -derivative \mathcal{F}'_S* of \mathcal{F} consists of all $\bar{x} = (x_1, x_2, \dots, x_l) \in X^{<\omega}$ for which there is a sequence $(y_i)_{i=1}^{\infty} \in S$ with $(\bar{x}, y_i) \in \mathcal{F}$ for all $i \in \mathbb{N}$.

Note that if \mathcal{F} is a hereditary tree then it follows that $\mathcal{F}'_S \subset \mathcal{F}$ and that \mathcal{F}'_S is also a hereditary tree (unless it is empty).

We then define higher order derivatives $\mathcal{F}_S^{(\alpha)}$ for ordinals $\alpha < \omega_1$ by recursion as follows.

$$\mathcal{F}_S^{(0)} = \mathcal{F}, \mathcal{F}_S^{(\alpha+1)} = (\mathcal{F}_S^{(\alpha)})'_S \text{ for } \alpha < \omega_1 \text{ and } \mathcal{F}_S^{(\lambda)} = \bigcap_{\alpha < \lambda} \mathcal{F}_S^{(\alpha)} \text{ for limit ordinals } \lambda < \omega_1.$$

It is clear that $\mathcal{F}_S^{(\alpha)} \supset \mathcal{F}_S^{(\beta)}$ if $\alpha \leq \beta$ and that $\mathcal{F}_S^{(\alpha)}$ is a hereditary tree (or the empty set) for all α , whenever \mathcal{F} is a hereditary tree. An easy induction also shows that

$$(\mathcal{F}(\bar{x}))_S^{(\alpha)} = (\mathcal{F}_S^{(\alpha)})(\bar{x}) \quad \text{for all } \bar{x} \in X^{<\omega}, \alpha < \omega_1.$$

We now define the S -index $I_S(\mathcal{F})$ of \mathcal{F} by

$$I_S(\mathcal{F}) = \min \{ \alpha < \omega_1 : \mathcal{F}_S^{(\alpha)} = \emptyset \}$$

if there exists $\alpha < \omega_1$ with $\mathcal{F}_S^{(\alpha)} = \emptyset$, and $I_S(\mathcal{F}) = \omega$ otherwise.

Remark. If λ is a limit ordinal and $\mathcal{F}_S^{(\alpha)} \neq \emptyset$ for all $\alpha < \lambda$, then in particular $\emptyset \in \mathcal{F}_S^{(\alpha)}$ for all $\alpha < \lambda$, and hence $\mathcal{F}_S^{(\lambda)} \neq \emptyset$. This shows that $I_S(\mathcal{F})$ is always a successor ordinal.

Examples 4.1. 1. A family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ can be thought of as a tree on \mathbb{N} : a set $F = \{m_1, \dots, m_k\} \in [\mathbb{N}]^{<\omega}$ is identified with $(m_1, \dots, m_k) \in \mathbb{N}^{<\omega}$ (recall that $m_1 < \dots < m_k$ by our convention of always listing the elements of a subset of \mathbb{N} in increasing order).

Let S be the set of all strictly increasing sequences in \mathbb{N} . In this case the S -index of a compact, family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is nothing else but the Cantor-Bendixson index of \mathcal{F} as a compact topological space, which we will continue to denote by $\text{CB}(\mathcal{F})$. We will also use the term Cantor-Bendixson derivative instead of S -derivative and use the notation \mathcal{F}' and $\mathcal{F}^{(\alpha)}$.

2. If X is an arbitrary set and $S = X^\omega$, then the S -index of a tree \mathcal{F} on X is what is usually called the *order of \mathcal{F}* (or the *height of \mathcal{F}*) denoted by $\text{o}(\mathcal{F})$. Note that in this case the S -derivative of \mathcal{F} consists of all finite sequences $\bar{x} \in X^{<\omega}$ for which there exists $y \in X$ such that $(\bar{x}, y) \in \mathcal{F}$.

The function $\text{o}(\cdot)$ is the largest index: for any $S \subset X^\omega$ we have $\text{o}(\mathcal{F}) \geq I_S(\mathcal{F})$.

We say that $S \subset X^\omega$ *contains diagonals* if for any sequence (\bar{x}_n) in S with $\bar{x}_n = (x_{n,i})_{i=1}^\infty$ there exist $i_1 < i_2 < \dots$ in \mathbb{N} so that $(x_{n,i_{n_k}})_{n=1}^\infty$ belongs to S . In particular every subsequence of every member of S belongs to S . If S contains diagonals, then the S -index of a tree on X may be measured via the Cantor-Bendixson index of the fine Schreier families $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$.

Proposition 4.2. [27, Proposition 5] *Let X be an arbitrary set and let $S \subset X^\omega$. If S contains diagonals, then for a hereditary tree \mathcal{A} on X and for a countable ordinal α the following are equivalent.*

- a) $\alpha < I_S(\mathcal{A})$.
- b) *There is a family $(x_F)_{F \in \mathcal{F}_\alpha \setminus \{\emptyset\}} \subset \mathcal{A}$ such that for $F = (m_1, m_2, \dots, m_k) \in \mathcal{F}_\alpha$ the branch $\bar{x}_F = (x_{\{m_1\}}, x_{\{m_1, m_2\}}, \dots, x_{\{m_1, m_2, \dots, m_k\}})$ is in \mathcal{A} , and $(x_{F \cup \{n\}})_{n > \max F}$ is in S , if F is not maximal in \mathcal{F}_α .*

Definition 4.3. Let $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ be regular, S a set of sequences in the set X , and and $(x_A : A \in \mathcal{F})$ a tree in X indexed by \mathcal{F} . We call $(x_A : A \in \mathcal{F})$ an S -tree if for every non maximal $A \in \mathcal{F}$ the sequence $(x_{A \cup \{n\}} : n \in \mathbb{N}, \text{ with } A \cup \{n\} \in \mathcal{F})$ is a sequence in S .

If X is a Banach space and S are the w -null sequences, we call $(x_A : A \in \mathcal{F})$ a w -null tree. Similarly we define w^* -null trees in X^* .

Remark 4.4. In the case of $X = \mathbb{N}$ and $S = [\mathbb{N}]^\omega$ we deduce from Proposition 4.2 that if $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ is hereditary and compact, then $\text{CB}(\mathcal{A}) > \alpha$ if and only if there is an order isomorphism $\pi : \mathcal{F}_\alpha \rightarrow \mathcal{A}$, so that for all $A \in \mathcal{F}_\alpha \setminus \text{MAX}(\mathcal{F}_\alpha)$ and $n > \max(A)$ it follows that $\pi(A \cup \{n\}) = \pi(A) \cup \{s_n\}$, where (s_n) is an increasing sequence in $\{s \in \mathbb{N} : s > \max \pi(A)\}$.

Examples 4.5. 1. *The weak index.* Let X be a separable Banach space. Let S be the set of all weakly null sequences in B_X , the unit ball of X . We call the S -index of a tree \mathcal{F} on X the *weak index of \mathcal{F}* and we shall denote it by $I_w(\mathcal{F})$. We shall use the term *weak derivative* instead of S -derivative and use the notation \mathcal{F}'_w and $\mathcal{F}_w^{(\alpha)}$.

When the dual space X^* is separable, the weak topology on the unit ball B_X , or on any bounded subset of X is metrizable. Hence in this case the set S contains diagonals and Proposition 4.2 applies.

2. *The weak* index.* We can define the weak* index similarly to the weak index. If X is a separable Banach space the w^* topology on B_{X^*} is metrizable. This implies that the set S of all w^* null sequences in B_{X^*} is diagonalizable. We call the S index of a tree \mathcal{F} on X^* *the weak* index of \mathcal{F}* and we shall denote it by $I_{w^*}(\mathcal{F})$. We shall use the term *weak* derivative* instead of S -derivative and use the notation \mathcal{F}'_{w^*} and $\mathcal{F}_{w^*}^{(\alpha)}$.

3. *The block index.* Let Z be a Banach space with an FDD $E = (E_i)$. A *block tree of (E_i) in Z* is a tree \mathcal{F} such that every element of \mathcal{F} is a (finite) block sequence of (E_i) . Let S be the set of all infinite block sequences of (E_i) in B_Z . We call the S -index of a block tree \mathcal{F} of (E_i) *the block index of \mathcal{F}* and we shall denote it by $I_{bl}(\mathcal{F})$. We shall use the term *block derivative* instead of S -derivative and use the notation \mathcal{F}'_{bl} and $\mathcal{F}_{bl}^{(\alpha)}$. Note that the set S contains diagonals, and hence Proposition 4.2 applies.

Note that (E_i) is a shrinking FDD of X if and only if every element of S is weakly null. In this case we have

$$(40) \quad I_{bl}(\mathcal{F}) \leq I_w(\mathcal{F})$$

for any block tree \mathcal{F} of (E_i) in Z . The converse is false in general, but it is true up to perturbations and without the assumption that (E_i) is shrinking.

5. THE SZLENK INDEX

Here we recall the definition and basic properties of the Szlenk index, and prove further properties that are relevant for our purposes.

Let X be a separable Banach space, and let K be a non-empty subset of X^* . For $\varepsilon \geq 0$ set

$$K'_\varepsilon = \left\{ x^* \in X^* : \exists (x_n^*) \subset K \text{ } w^* - \lim_{n \rightarrow \infty} x_n^* = x^* \text{ and } \|x_n^* - x^*\| > \varepsilon \right\}.$$

and define $K_\varepsilon^{(\alpha)}$ for each countable ordinal α by recursion as follows:

$$K_\varepsilon^{(0)} = K, K_\varepsilon^{(\alpha+1)} = (K_\varepsilon^{(\alpha)})'_\varepsilon \text{ for } \alpha < \omega_1, \text{ and } K_\varepsilon^{(\lambda)} = \bigcap_{\alpha < \lambda} K_\varepsilon^{(\alpha)} \text{ for limit ordinals } \lambda < \omega_1.$$

Next, we associate to K the following ordinal indices:

$$\eta(K, \varepsilon) = \sup\{\alpha < \omega_1 : K_\varepsilon^{(\alpha)} \neq \emptyset\}, \text{ and } \eta(K) = \sup_{\varepsilon > 0} \eta(K, \varepsilon).$$

Finally, we define the *Szlenk index* $Sz(X)$ of X to be $\eta(B_{X^*})$, where B_{X^*} is the unit ball of X^* .

Remark. The original definition of the derived sets K'_ε in [35] is slightly different. It might lead to different values of $Sz(K, \varepsilon)$, but to same values of $Sz(K)$, and in particular to the same values of $Sz(X)$.

Szlenk used his index to show that there is no separable, reflexive space universal for the class of all separable, reflexive spaces. This result follows immediately from the following properties of the function $Sz(\cdot)$.

Theorem 5.1. [35] *Let X and Y be separable Banach spaces.*

- (i) X^* is separable if and only if $Sz(X) < \omega_1$.

- (ii) If X isomorphically embeds into Y , then $\text{Sz}(X) \leq \text{Sz}(Y)$.
- (iii) For all $\alpha < \omega_1$ there exists a separable, reflexive space with Szlenk index at least α .

We also recall the following observation of [2] about the form of the Szlenk index of a Banach space with separable dual.

Proposition 5.2. [2, Theorems 3.22 and 4.2] *If X has a separable dual then there is an $\alpha < \omega_1$ with $\text{Sz}(X) = \omega^\alpha$.*

The following Theorem combines several equivalent description of the Szlenk index of a separable space not containing ℓ_1

Theorem 5.3. *Assume that X is a separable space not containing ℓ_1 and $\alpha < \omega_1$. The following conditions are equivalent.*

- (i) $\text{Sz}(X) > \omega^\alpha$.
 - (ii) There is an $\varepsilon > 0$ and a tree $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_{X^*}$, so that for any non maximal $A \in \mathcal{S}_\alpha$ we have
- $$(41) \quad w^* - \lim_{n \rightarrow \infty} z_{A \cup \{n\}}^* = z_A^* \text{ and } \|z_A^* - z_{A \cup \{n\}}^*\| > \varepsilon \text{ for } n > \max(A).$$
- (iii) There is an $\varepsilon > 0$, a tree $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_X^*$ and a w -null tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, so that

$$(42) \quad z_B^*(z_A) > \varepsilon, \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \preceq B,$$

$$(43) \quad |z_B^*(z_A)| < \varepsilon/2 \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \not\preceq B,$$

and for all non maximal $A \in \mathcal{S}_\alpha$ we have

$$(44) \quad w^* - \lim_{n \rightarrow \infty} z_{A \cup \{n\}}^* = z_A^*.$$

- (iv) There is an $N \in [\mathbb{N}]^\omega$, an $\varepsilon > 0$ and a w -null tree $(x_A : A \in \mathcal{S}_\alpha \cap [N]^{<\omega})$, so that for every maximal B in $\mathcal{S}_\alpha \cap [N]^{<\omega}$ we have

$$\left\| \sum_{A \preceq B} \zeta(\alpha, A) x_A \right\| \geq \varepsilon.$$

- (v) There is an $\varepsilon > 0$, so that $I_w(\mathcal{F}_\varepsilon) > \omega^\alpha$, where

$$\mathcal{F}_\varepsilon = \left\{ (x_1, x_2, \dots, x_l) \subset S_X : \forall (a_j)_{j=1}^l \subset [0, 1] \quad \left\| \sum_{j=1}^l a_j x_j \right\| \geq \varepsilon \sum_{j=1}^l a_j \right\}.$$

- (vi) There is an $\varepsilon > 0$ so that $I_{w^*}(\mathcal{G}_\varepsilon) > \omega^\alpha$, where

$$\mathcal{G}_\varepsilon = \left\{ (x_1^*, x_2^*, \dots, x_l^*) \subset B_{X^*} : \|x_j^*\| \geq \varepsilon \text{ and } \left\| \sum_{i=1}^j x_i^* \right\| \leq 1, \text{ for } j = 1, 2, \dots, l \right\}.$$

Proof. In order to show “(i) \Rightarrow (ii)” we first prove the following

Lemma 5.4. *Let X be a separable Banach space and $K \subset X^*$ be w^* -compact, $0 < \varepsilon < 1$, and $\beta < \omega_1$. Then for every $x^* \in K_\varepsilon^{(\beta)}$ there is a family $(z_{(x^*, A)}^* : A \in \mathcal{F}_\beta) \subset K$, so that*

$$(45) \quad z_{(x^*, A)}^* \in K \text{ and } z_{(x^*, \emptyset)}^* = x^*,$$

$$(46) \quad \text{if } A \text{ is not maximal in } \mathcal{F}_\beta, \text{ then } \|z_{(x^*, A)}^* - z_{(x^*, A \cup \{n\})}^*\| > \varepsilon \text{ for all } n > \max(A),$$

$$(47) \quad \text{if } A \text{ is not maximal in } \mathcal{F}_\beta, \text{ then } z_{(x^*, A)}^* = w^* - \lim_{n \rightarrow \infty} z_{(x^*, A \cup \{n\})}^*.$$

Proof. We will prove our claim by transfinite induction for all $\beta < \omega_1$. Let us first assume that $\beta = 1$. For $x^* \in K'_\varepsilon$ choose a sequence (x_n^*) which w^* -converges to x^* , with $\|x^* - x_n^*\| > \varepsilon$, for $n \in \mathbb{N}$. Thus, we can choose $z_{(x^*, \emptyset)} = x^*$, and $z_{(x^*, \{n\})} = x_n^*$. This choice satisfies (45), (46), and (47), for $\beta = 1$ (recall that $\mathcal{F}_1 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$).

Now assume that our claim is true for all $\gamma < \beta$. First assume that β is a successor ordinal and let $\gamma < \omega_1$ so that $\beta = \gamma + 1$. Let $x^* \in K_\varepsilon^{(\gamma+1)}$. Thus there is a sequence $(x_n^*) \subset K_\varepsilon^{(\gamma)}$ which w^* -converges to x^* , with $\|x_n^* - x^*\| > \varepsilon$, for $n \in \mathbb{N}$. By our induction hypothesis we can choose for each $n \in \mathbb{N}$ a family $(z_{(x_n^*, A)}^* : A \in \mathcal{F}_\gamma)$ satisfying (45), (46), and (47), for γ and x_n^* instead of x^* . For every $A \in \mathcal{F}_{\gamma+1}$ it follows that $A \setminus \{\min A\} \in \mathcal{F}_\gamma$ and we define $z_{(x^*, \emptyset)}^* := x^*$ and for $A \in \mathcal{F}_{\gamma+1} \setminus \{\emptyset\}$

$$z_{(x^*, A)}^* := z_{(x_{\min A}^*, A \setminus \{\min(A)\})}^*.$$

It is then easy to see that $(z_{(x^*, A)}^* : A \in \mathcal{F}_{\gamma+1})$ satisfies (45), (46), and (47).

Assume that $\beta < \omega_1$ is a limit ordinal and let $(\mu(\beta, n) : n \in \mathbb{N}) \subset (0, \beta)$, be the sequence of ordinals increasing to β , used to define \mathcal{F}_β . We abbreviate $\beta_n = \mu(\beta, n)$, for $n \in \mathbb{N}$. Let $x^* \in K_\varepsilon^{(\beta)} = \bigcap_{\gamma < \beta} K_\varepsilon^{(\gamma)}$. Since $\beta_n + 1 < \beta$, we can use for each $n \in \mathbb{N}$ our induction hypothesis and choose a family

$$(z_{(n, x^*, A)}^* : A \in \mathcal{F}_{\beta_n+1}) \subset X^*,$$

satisfying (45), (46), and (47), for $\beta_n + 1$. In particular it follows that

$$w^* - \lim_{j \rightarrow \infty} z_{(n, x^*, \{j\})}^* = x^*, \text{ for all } n \in \mathbb{N}.$$

Since the w^* -topology is metrizable on K we can find an increasing sequence $(j_n : n \in \mathbb{N})$ in \mathbb{N} , $j_n > n$, for $n \in \mathbb{N}$, so that

$$w^* - \lim_{n \rightarrow \infty} z_{(n, x^*, \{j_n\})}^* = x^*.$$

Consider for $n \in \mathbb{N}$ the set

$$\mathcal{F}_{\beta_n+1}(j_n) = \{A \in [\mathbb{N}]^{<\omega} : j_n < \min A \text{ and } \{j_n\} \cup A \in \mathcal{F}_{\beta_n+1}\} = \{A \in \mathcal{F}_{\beta_n} : j_n < \min A\}.$$

Since \mathcal{F}_{β_n} is spreading, for $n \in \mathbb{N}$, we can choose $L_n = \{l_1^{(n)}, l_2^{(n)}, \dots\} \in [\mathbb{N}]^\omega$ so that

$$\mathcal{F}_{\beta_n}^{L_n} = \{\{l_{a_1}^{(n)}, l_{a_2}^{(n)}, \dots, l_{a_m}^{(n)}\} : \{a_1, a_2, \dots, a_m\} \in \mathcal{F}_{\beta_n}\} \subset \mathcal{F}_{\beta_n+1}(j_n).$$

We define the map

$$\phi_n : \mathcal{F}_{\beta_n} \rightarrow \mathcal{F}_{\beta_n+1}(j_n), \{a_1, a_2, \dots, a_m\} \mapsto \{l_{a_1}^{(n)}, l_{a_2}^{(n)}, \dots, l_{a_m}^{(n)}\}.$$

Then we put for $A \in \mathcal{F}_\beta$

$$z_{(x^*, A)}^* = \begin{cases} x^* & \text{if } A = \emptyset, \\ z_{(x^*, \{j_n\})}^* & \text{if } A = \{n\} \text{ for some } n \in \mathbb{N}, \text{ and} \\ z_{(x^*, \{j_n\} \cup \phi_n(B))}^* & \text{if } A = \{n\} \cup B \text{ for some } n \in \mathbb{N} \text{ and } B \in \mathcal{F}_{\alpha_n} \setminus \{\emptyset\}, \end{cases}$$

which has the wanted property also in that case. \square

We now continue with our proof of “(i) \Rightarrow (ii)” of Theorem 5.3. Assuming now that $\text{Sz}(X) > \beta = \omega^\alpha$, it follows that $[B_{X^*}]_\varepsilon^{(\beta)} \neq \emptyset$ for some $\varepsilon > 0$. We choose $x^* \in [B_{X^*}]_\varepsilon^{(\beta)}$ and apply Lemma 5.4 to obtain a tree in B_{X^*} indexed by $\mathcal{F}_{\omega^\alpha}$ satisfying the conditions (45), (46), and (47). Now Proposition 2.12 and a relabeling of the tree, yields (ii).

“(ii) \Rightarrow (iii)” For $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ we define $A' = A \setminus \{\max(A)\}$. Now let $(A_m)_{m \in \mathbb{N}}$ be a consistent ordering of \mathcal{S}_α (see Subsection 2.4). We write $A <_{lin} B$ or $A \leq_{lin} B$ if $A = A_m$ and $B = A_n$, for $m < n$ or $m \leq n$, respectively.

Let $\varepsilon > 0$ and $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_{X^*}$, so that (41) is satisfied. Then choose for each $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ an element $x_A \in S_X$ so that $(z_A^* - z_{A'}^*)(x_A) > \varepsilon$.

Let $0 < \eta < \varepsilon/8$, and let $(\eta(A) : A \in \mathcal{S}_\alpha) \subset (0, 1)$ satisfy the following conditions

$$(48) \quad (\eta(A)) \text{ is decreasing with respect to the linear ordering } <_{lin},$$

$$(49) \quad \sum_{A \in \mathcal{S}_\alpha} \eta(A) < \eta,$$

$$(50) \quad \sum_{B \in \mathcal{S}_\alpha, B >_{lin} A} \eta(B) < \eta(A), \text{ for all } A \in \mathcal{S}_\alpha, \text{ and}$$

$$(51) \quad \eta(A_m) < \frac{1}{2} \frac{\eta}{m+2}, \text{ for all } m \in \mathbb{N}.$$

Since X does not contain a copy of ℓ_1 we can apply Rosenthal's ℓ_1 Theorem, and assume, possibly after passing to a pruning, that for each non maximal $A \in \mathcal{S}_\alpha$ the sequence $(x_{A \cup \{n\}})_{n > \max(A)}$ is weakly Cauchy. Since $(z_{A \cup \{n\}}^* - z_A^*)_{n > \max(A)}$ is w^* -null we can assume, possibly after passing to a further pruning, that $(z_{A \cup \{n\}}^* - z_A^*)(x_{A \cup \{n-1\}}) < \eta(A \cup \{n\})$, for all non maximal $A \in \mathcal{S}_\alpha$ and $n > 1 + \max(A)$.

Let $z_\emptyset = 0$. For a non maximal element $A \in \mathcal{S}_\alpha$ and $n > 1 + \max(A)$ let

$$z_{A \cup \{n\}} = \frac{1}{2}(x_{A \cup \{n\}} - x_{A \cup \{n-1\}})$$

and

$$z_{A \cup \{\max(A)+1\}} = x_{A \cup \{\max(A)+1\}}.$$

Then the families $(z_A : A \in \mathcal{S}_\alpha)$ and $(z_A^* : A \in \mathcal{S}_\alpha)$ are in B_X and B_{X^*} respectively, $(z_A : A \in \mathcal{S}_\alpha)$ is weakly null and $(z_A^* : A \in \mathcal{S}_\alpha)$ satisfies (44).

Moreover it follows that

$$(52) \quad (z_A^* - z_{A'}^*)(z_A) \geq \frac{\varepsilon}{2} - \eta(A), \text{ for all } A \in \mathcal{S}_\alpha \setminus \{\emptyset\}.$$

Since $w - \lim_{n \rightarrow \infty} z_{B \cup \{n\}} = 0$ and $w^* - \lim_{n \rightarrow \infty} z_{B \cup \{n\}}^* = z_B^*$, for every non maximal $B \in \mathcal{S}_\alpha$ we can, after passing again to a pruning, assume that

$$(53) \quad |(z_B^* - z_{B'}^*)(z_A)| < \eta(B) \text{ and } |z_A^*(z_B)| < \eta(B) \text{ for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{lin} B.$$

We are left with verifying (42) and (43) for $\varepsilon/4$ instead of ε .

To show (42) let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \preceq B$. We choose $l \in \mathbb{N}$ and $A = B_0 \prec B_1 \prec B_2 \prec \dots \prec B_l = B$, so that $B'_j = B_{j-1}$, for $j = 1, 2, \dots, l$, and deduce from (52) and (53)

$$\begin{aligned} z_B^*(z_A) &= \sum_{j=1}^l (z_{B_j}^* - z_{B'_j}^*)(z_A) + (z_A^* - z_{A'}^*)(z_A) + z_{A'}^*(z_A) \\ &\geq \frac{\varepsilon}{2} - \sum_{j=1}^l \eta(B_j) - 2\eta(A) > \frac{\varepsilon}{2} - 2\eta > \frac{\varepsilon}{4}. \end{aligned}$$

In order to show (43) let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \not\preceq B$. We choose $l \in \mathbb{N}$ and $\emptyset = B_0 \prec B_1 \prec B_2 \prec \dots \prec B_l = B$ so that $B'_j = B_{j-1}$, for $j = 1, 2, \dots, l$ and, since for every

$j = 1, 2, \dots, l$, either $A <_{lin} B_j$ or $B_j <_{lin} A$ we deduce from (53) and the conditions (49) and (51) on $\eta(\cdot)$, that

$$\begin{aligned} |z_B^*(z_A)| &\leq \left| \sum_{j=1}^l (z_{B_j}^* - z_{B_j'}^*)(z_A) \right| + |z_\emptyset^*(z_A)| \\ &\leq \sum_{A <_{lin} B_j} |(z_{B_j}^* - z_{B_j'}^*)(z_A)| + \sum_{A >_{lin} B_j} (|z_{B_j}^*(z_A)| + |z_{B_{j-1}}^*(z_A)|) + \eta(A) \\ &\leq \sum_{A <_{lin} B_j} \eta(B_j) + 2 \sum_{B_j <_{lin} A} \eta(A) + \eta(A) \leq (2\#\{j \leq l : B_j <_{lin} A\} + 2)\eta(A) < \frac{\varepsilon}{4} \end{aligned}$$

which verifies (43) and finishes the proof of our claims.

“(iii) \Rightarrow (iv)” Let $\varepsilon > 0$, $(z_A^* : A \in \mathcal{S}_\alpha)$, and $(z_A : A \in \mathcal{S}_\alpha)$ satisfy the condition in (iii). Then it follows for a maximal $B \in \mathcal{S}_\alpha$ that

$$\left\| \sum_{A \preceq B} \zeta(\alpha, A) z_A \right\| \geq \sum_{A \preceq B} \zeta(\alpha, A) z_B^*(z_A) \geq \varepsilon \sum_{A \preceq B} \zeta(\alpha, A) = \varepsilon,$$

which proves our claim.

“(iv) \Rightarrow (v)” Assume that $N \in [\mathbb{N}]^\omega$, $\varepsilon > 0$, and $(x_A : A \in \mathcal{S}_\alpha \cap [N]^{<\omega}) \subset B_X$ satisfy (iv). For $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ put $y_B = \sum_{A \prec B} \zeta(\alpha, A) x_A$, and choose $y_B^* \in S_{X^*}$ so that $y_B^*(y_B) = \|y_B\| > \varepsilon$.

For $B \in \text{MAX}(\mathcal{S}_\alpha) \cap [N]^{<\omega}$, we define $f_B : B \rightarrow [-1, 1]$, $b \mapsto y_B^*(x_{\{a \in B, a \leq b\}})$. From Corollary 3.5 it follows now for $\delta = \varepsilon/2$ that $\text{CB}(\mathcal{A}_{\delta, N}) = \omega^\alpha + 1$ where

$$\mathcal{A}_{\delta, N} = \left\{ A \in \mathcal{S}_\alpha \cap [N]^{<\omega} : \begin{array}{l} \exists B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega}), \\ A \subset B, \text{ and } f_B(a) \geq \delta \text{ for all } a \in A \end{array} \right\}.$$

But from Proposition 4.2 and the Remark thereafter we deduce that there is a order isomorphism $\pi : \mathcal{F}_{\omega^\alpha} \rightarrow \mathcal{A}_{\delta, N}$ so that for every non maximal $A \in \mathcal{F}_{\omega^\alpha}$ and any $n > \max(A)$ it follows that $\pi(A \cup \{n\}) = \pi(A) \cup \{s_n\}$, for some increasing sequence $(s_n) \subset \mathbb{N}$. Putting $z_A = x_{\pi(A)}$ it follows that $(z_A)_{A \in \mathcal{F}_{\omega^\alpha}}$ is a weakly null tree and for every $A = \{a_1, a_2, \dots, a_l\}$ it follows that $(z_{\{a_1, a_2, \dots, a_l\}})_{l=1}^\infty \in \mathcal{F}_\delta$. Applying again Proposition 4.2 yields (v).

“(v) \iff (i)” This follows from [2, Theorem 4.2] where it was shown that $\text{Sz}(X) = \sup_{\varepsilon > 0} I_w(\mathcal{F}_\varepsilon)$, if ℓ_1 does not embed into X .

“(ii) \iff (vi)” Follows from Proposition 2.13 and an application of Proposition 4.2 to the tree \mathcal{G}_ε on B_{X^*} , and $S = \{(x_n^*) \subset B_X^* : w^* - \lim_{n \rightarrow \infty} = 0\}$. \square

Remark. We note that in the implication (i) \Rightarrow (ii) the assumption that ℓ_1 does not embed into X was not needed.

We will also need the following *dual version* of Theorem 5.3.

Proposition 5.5. *Assume that X is a Banach space whose dual X^* is separable, with $\text{Sz}(X^*) > \omega^\alpha$. Then there is an $\varepsilon > 0$, a tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, and a w^* -null tree $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_X^*$ so that*

$$(54) \quad z_A^*(z_B) > \varepsilon, \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \preceq B,$$

$$(55) \quad |z_A^*(z_B)| < \frac{\varepsilon}{2} \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \not\preceq B.$$

Proof. Recall that, as stated above, the implication (i) \Rightarrow (ii) of Theorem 5.3 holds even if the space, to which the theorem is applied, contains ℓ_1 . Applying this implication to X^* , we find $\varepsilon > 0$ and $\{z_A^{**} : A \in \mathcal{S}_\alpha\} \subset B_{X^{**}}$, so that (41) is satisfied. Then choose for each $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$ an element $x_A^* \in S_X$ so that $(z_A^{**} - z_{A'}^{**})(x_A^*) > \varepsilon$. Again let (A_n) be a consistent enumeration of \mathcal{S}_α , and write $A_m <_{\text{lin}} A_n$ if $m < n$. We also assume that $(\eta(A) : A \in \mathcal{S}_\alpha) \subset (0, 1)$, has the property that

$$(56) \quad \sum_{A \in \mathcal{S}_\alpha} \eta(A) < \frac{\varepsilon}{32}.$$

After passing to a first pruning we can assume that for all non maximal $A \in \mathcal{S}_\alpha$ the sequence $(x_{A \cup \{n\}}^*)$ w^* -converges, and that for any $B \in \mathcal{S}_\alpha$ the sequence $z_B^{**}(x_{A \cup \{n\}}^*)$ converges to some number $r_{A,B}$ (for fixed $A, B \in \mathcal{S}_\alpha$ we only need to pass to a subsequence of $(A \cup \{n\} : n \in \mathbb{N}, A \cup \{n\} >_{\text{lin}} B)$). Since $(z_{A \cup \{n\}}^{**} - z_A^{**})_{n > \max(A)}$ is w^* -null we can assume, after passing to a second pruning, that we have

$$(57) \quad |(z_B^{**} - z_{B'}^{**})(x_A^*)| < \eta(B) \text{ for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{\text{lin}} B,$$

We put $z_\emptyset^* = 0$ and for any non maximal element $A \in \mathcal{S}_\alpha$

$$z_{A \cup \{\max(A)+1\}}^* = x_{A \cup \{\max(A)+1\}}^* \text{ and } z_{A \cup \{n\}}^* = \frac{1}{2}(x_{A \cup \{n\}}^* - x_{A \cup \{n-1\}}^*) \text{ if } n > \max(A) + 1.$$

It follows that $(z_A^* : A \in \mathcal{S}_\alpha)$ is a w^* -null tree in B_{X^*} and that for any $A \in \mathcal{S}_\alpha$

$$(z_A^{**} - z_{A'}^{**})(z_A^*) > \frac{\varepsilon}{2} - \frac{\eta(A)}{2}.$$

Since $z_B^{**}(x_{A \cup \{n\}}^*)$ converges to $r_{A,B}$ we can assume, after passing to a third pruning, that

$$(58) \quad |z_A^{**}(z_B^*)| < \frac{\eta(B)}{2} \text{ whenever } A <_{\text{lin}} B.$$

and hence

$$(59) \quad z_A^{**}(z_A^*) > \frac{\varepsilon}{2} - \eta(A) \text{ for all } A \in \mathcal{S}_\alpha.$$

Since $w^*\text{-}\lim_n z_{A \cup \{n\}}^{**} = z_A^{**}$ we can assume, by passing to a further pruning, that

$$(60) \quad |(z_B^{**} - z_{B'}^{**})(z_A^*)| < \eta(B) \text{ for all } A, B \in \mathcal{S}_\alpha, \text{ with } A <_{\text{lin}} B.$$

Since B_X is w^* -dense in $B_{X^{**}}$ we can choose, for every $A \in \mathcal{S}_\alpha$, a vector $z_A \in B_X$ so that

$$(61) \quad |z_A^*(z_B) - z_B^{**}(z_A^*)| < \eta(B), \text{ for all } A, B \in \mathcal{S}_\alpha, \text{ with } A \leq_{\text{lin}} B.$$

Combining (60) and (61) we obtain that for all A and B in \mathcal{S}_α , with $A <_{\text{lin}} B$ we have

$$(62) \quad |z_A^*(z_B) - z_{B'}^{**}(z_A^*)| \leq |z_A^*(z_B) - z_B^{**}(z_A^*)| + |(z_B^{**} - z_{B'}^{**})(z_A^*)| < 2\eta(B).$$

Using that $(z_A^* : A \in \mathcal{S}_\alpha)$ is a w^* -null tree, we can pass to a further pruning, so that

$$(63) \quad |z_B^*(z_A)| < \eta(B), \text{ for all } A, B \in \mathcal{S}_\alpha \text{ with } A <_{\text{lin}} B.$$

We deduce from (60) and (61), for $A, B \in \mathcal{S}_\alpha$, with $A \leq_{\text{lin}} B' (<_{\text{lin}} B)$ that

$$(64) \quad |z_A^*(z_B - z_{B'})| \leq |z_A^*(z_B) - z_{B'}^{**}(z_A^*)| + |z_A^*(z_{B'}) - z_{B'}^{**}(z_A^*)| + |(z_B^{**} - z_{B'}^{**})(z_A^*)| \\ \leq 2\eta(B) + \eta(B').$$

By (62), (58), and (63) for $A, B \in \mathcal{S}_\alpha$, with $B' <_{\text{lin}} A < B$ we obtain

$$(65) \quad |z_A^*(z_B - z_{B'})| \leq |z_A^*(z_B) - z_{B'}^{**}(z_A^*)| + |z_{B'}^{**}(z_A^*)| + |z_A^*(z_{B'})| \leq 2\eta(B) + 2\eta(A).$$

We now claim that the families $\{z_A : A \in \mathcal{S}_\alpha\}$ and $\{z_A^* : A \in \mathcal{S}_\alpha\}$ satisfy (54) and (55).

In order to verify (54), let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \preceq B$. Then let $k \in \mathbb{N}$ and $B_j \in \mathcal{S}_\alpha$, for $j = 0, 1, 2, \dots, k$, be such that $A = B_0 \prec B_1 \prec B_2 \prec \dots \prec B_k = B$, and $B'_j = B_j$, for $j = 1, 2, \dots, k$.

$$\begin{aligned} z_A^*(z_B) &= \sum_{j=1}^k z_A^*(z_{B_j} - z_{B'_j}) + z_A^*(z_A) \\ &\geq z_A^{**}(z_A^*) - |z_A^{**}(z_A^*) - z_A^*(z_A)| - |z_A^*(z_{B_1} - z_A)| - \sum_{j=2}^k |z_A^*(z_{B_j} - z_{B'_j})| \\ &> \frac{\varepsilon}{2} - 3\eta(A) - 2 \sum_{j=1}^k \eta(B_j) \geq \frac{\varepsilon}{4}. \quad (\text{By (59), (61), (64) and (62)}) \end{aligned}$$

which yields (54) if we replace ε by $\varepsilon/4$.

In order to verify (55), let $A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, with $A \not\preceq B$. If $A >_{\text{lin}} B$ we deduce our claim from (63). If $A \leq_{\text{lin}} B$, and thus $A <_{\text{lin}} B$, we choose $k \in \mathbb{N}$ and $B_0 \prec B_1 \prec B_2 \prec \dots \prec B_k = B$, with $B'_j = B_{j-1}$, for $j = 1, 2, \dots, k$, and $B_0 <_{\text{lin}} A <_{\text{lin}} B_1$. Applying (65), (63), (64), and finally (56) we obtain

$$\begin{aligned} |z_A^*(z_B)| &\leq |z_A^*(z_{B_1} - z_{B_0})| + |z_A^*(z_{B_0})| + \left| \sum_{j=2}^k z_A^*(z_{B_j} - z_{B'_j}) \right| \\ &\leq 2\eta(B_1) + \eta(B_0) + 3\eta(A) + \sum_{j=2}^k (2\eta(B_j) + \eta(B_{j-1})) \leq \frac{\varepsilon}{8} \end{aligned}$$

which proves our claim. \square

Example 5.6. Let us construct an example of families $(z_A : A \in [\mathbb{N}]^{<\omega}) \subset B_{c_0}$ and $(z_A^* : A \in [\mathbb{N}]^{<\omega}) \subset B_{\ell_1}$ satisfying Proposition 5.5. Let $<_{\text{lin}}$ again be a linear consistent ordering of $[\mathbb{N}]^{<\omega}$. We first choose a family $(\tilde{A} : A \in [\mathbb{N}]^{<\omega}) \subset [\mathbb{N}]^{<\omega}$ with the following properties

- (66) \tilde{A} is a spread of A , for each $A \in [\mathbb{N}]^{<\omega}$,
- (67) $A \prec B$ if and only if $\tilde{A} \prec \tilde{B}$,
- (68) if $A, B \in [\mathbb{N}]^{<\omega}$, $\emptyset \neq A <_{\text{lin}} B$, and $C \in [\mathbb{N}]^{<\omega}$ is the maximal element in $[\mathbb{N}]^{<\omega}$, so that $C \preceq A$ and $C \preceq B$, then $(\tilde{A} \setminus \tilde{C}) \cap (\tilde{B} \setminus \tilde{C}) = \emptyset$.

We define for $A \in [\mathbb{N}]^{<\omega}$

$$z_A = \sum_{a \in \tilde{A}} e_a \text{ and } z_A^* = e_{\max(\tilde{A})}^*,$$

where (e_j) and (e_j^*) denote the unit vector bases in c_0 and ℓ_1 respectively. It is now easy to verify that the tree (z_A^*) is w^* -null and that (54) is satisfied for any $\varepsilon \in (0, 1)$. In order to verify (55) let $A, B \in [\mathbb{N}]^{<\omega}$ with $A \not\preceq B$. If $A >_{\text{lin}} B$, then $\max(\tilde{A}) \notin \tilde{B}$, and our claim follows. If $A <_{\text{lin}} B$ let $C \in [\mathbb{N}]^{<\omega}$ be the maximal element for which $C \preceq A$ and $C \preceq B$. It follows that $C \prec A$, but also that $C \prec B$, which implies by (68) that $\max(\tilde{A}) \notin \tilde{B}$, and, thus our claim.

6. ESTIMATING CERTAIN CONVEX COMBINATIONS OF BLOCKS USING THE SZLENK INDEX

In this section we will assume that X has an FDD (F_j) . This means that $F_j \subset X$ is a finite dimensional subspace of X , for $j \in \mathbb{N}$, and that every x has a unique representation as sum $x = \sum_{j=1}^{\infty} x_j$, with $x_j \in F_j$, for $j \in \mathbb{N}$. For $x = \sum_{j=1}^{\infty} x_j \in X$ we call $\text{supp}(x) = \{j \in \mathbb{N} : x_j \neq 0\}$ the *support of x* (with respect to (F_j)), and the smallest interval in \mathbb{N} containing $\text{supp}(x)$ is called the *range of x* (with respect to (F_j)), and is denoted by $\text{ran}(x)$. A (finite or infinite) sequence $(x_n) \subset X$ is called a *block (with respect to (F_j))*, if $x_n \neq 0$, for all $n \in \mathbb{N}$, and $\text{supp}(x_n) < \text{supp}(x_{n+1})$, for all $n \in \mathbb{N}$ for which x_{n+1} is defined.

We call an FDD *shrinking* if every bounded block $(x_n)_{n=1}^{\infty}$ is weakly null. As in the case of bases, X^* is separable, and thus $\text{Sz}(X) < \omega_1$, if X has a shrinking FDD.

Theorem 6.1. *Let X be a Banach space with a shrinking FDD and α be a countable ordinal number with $\text{Sz}(X) \leq \omega^\alpha$. Then for every $\varepsilon > 0$ and $M \in [\mathbb{N}]^\omega$, there exists $N \in [M]^\omega$ satisfying the following: for every $B = \{b_1, \dots, b_d\}$ in $\mathcal{S}_\alpha \cap [N]^{<\omega}$ and sequence $(x_i)_{i=1}^d$ in B_X , with $\text{ran}(x_j) \subset (b_{j-1}, b_{j+1})$ for $j = 1, \dots, d$ (where $b_0 = 0$ and $b_{d+1} = \infty$), we have*

$$(69) \quad \left\| \sum_{j=1}^d \zeta(\alpha, B_j) x_j \right\| < \varepsilon,$$

where $B_j = \{b_1, \dots, b_j\}$ for $j = 1, \dots, d$.

Proof. It is enough to find N in $[M]^\omega$ so that (69) holds whenever $B \in \text{MAX}(\mathcal{S}_\alpha) \cap [N]^{<\omega}$. Indeed, if (69) holds for all B in $\text{MAX}(\mathcal{S}_\alpha) \cap [N]^{<\omega}$, then for any $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ and $(x_i)_{i=1}^{\#A}$ satisfying the assumption of Theorem 6.1, one may extend A to any maximal set B and extend the sequence $(x_k)_{k=1}^{\#A}$ by concatenating the zero vector $\#B - \#A$ times. Towards a contradiction, we assume that such a set N does not exist. Applying Proposition 2.14 to the Partition $(\mathcal{F}, \mathcal{S}_\alpha \setminus \mathcal{F})$ of \mathcal{S}_α , where

$$\mathcal{F} = \left\{ B = \{b_1, b_2, \dots, b_n\} \in \text{MAX}(\mathcal{S}_\alpha) : \begin{array}{l} \exists (x_j) \subset B_X, \text{ran}(x_j) \subset (b_{j-1}, b_{j+1}), \text{ for } j=1, 2, \dots, n \\ \left\| \sum_{j=1}^n \zeta(\alpha, \{b_1, b_2, \dots, b_j\}) x_j \right\| > \varepsilon \end{array} \right\},$$

yields that there is L in $[M]^\omega$, so that for all $B = \{b_1^B, \dots, b_{d_B}^B\}$ in $\text{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$, there exists a sequence $(x_i^B)_{i=1}^{d_B}$ in B_X with $\text{ran}(x_j) \subset (b_{j-1}^B, b_{j+1}^B)$ for $j = 1, \dots, d_B$, so that

$$(70) \quad \left\| \sum_{j=1}^{d_B} \zeta(\alpha, B_j^B) x_j^B \right\| \geq \varepsilon,$$

where $B_j^B = \{b_1^B, \dots, b_j^B\}$ for $j = 1, \dots, d_B$. For $A \preceq B$, if $A = B_j^B$, we use the notation $x_j^B = x_A^B$. Note that, under this notation, (70) takes the more convenient form

$$(71) \quad \left\| \sum_{A \preceq B} \zeta(\alpha, A) x_A^B \right\| \geq \varepsilon$$

and that

$$(72) \quad \text{ran}(x_{A'}^B) \subset (\max(A''), \max(A)), \text{ for all } A \preceq B \text{ with } A' \neq \emptyset,$$

where $A'' = (A')'$ and $\max(\emptyset) = 0$.

We will now apply several stabilization and perturbation arguments to show that we may assume that for $B \in \text{MAX}(\mathcal{S}_\alpha)$ and $A \preceq B$ the vector $x_{A'}^B$, only depends on A , and will be renamed x_A .

Using a compactness argument, Proposition 2.14 yields the following: if $A \in \mathcal{S}_\alpha \cap [L]^{<\omega}$ is non-maximal with $A' \neq \emptyset$, then for $\delta > 0$, there is $L' \in [L]^\omega$ so that for all D_1, D_2 in $\text{MAX}(\mathcal{S}_\alpha(A)) \cap [L']^{<\omega}$, we have $\|x_{A'}^{A \cup D_1} - x_{A'}^{A \cup D_2}\| < \delta$. Combining the above with a standard diagonalization argument we may pass to a further infinite subset of L , and a perturbation of the block vectors $x_{A'}^B$ with $B \in \text{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$ and $\{\min(B)\} \prec A \preceq B$ (and perhaps pass to a smaller ε in (71)), so that for every B_1, B_2 in $\text{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$ and A , with $A' \neq \emptyset$, so that $A \preceq B_1$ and $A \preceq B_2$, we have $x_{A'}^{B_1} = x_{A'}^{B_2}$. For every $A \in \mathcal{S}_\alpha \cap [L]^{<\omega}$, we call this common vector x_A . Note that x_A indeed depends on A and not only on A' . For A so that $A' = \emptyset$, i.e. for those sets A that are of the form $A = \{n\}$ for some $n \in L$, chose any normalized vector x_A with $\text{supp}(x_A) = \{n\}$. Note that, using (72), we have

$$(73) \quad \text{ran}(x_A) \subset (\max(A''), \max(A)) \text{ for all } A \in \mathcal{S}_\alpha \cap [L]^{<\omega} \text{ with } A' \neq \emptyset,$$

where $\max(\emptyset) = 0$ and if $A' = 0$, i.e. $A = \{n\}$ for some $n \in \mathbb{N}$, then $\text{ran}(x_A) = \{n\}$.

Furthermore, fixing $0 < \delta < \varepsilon/12$ and passing to an infinite subset of L , again denoted by L , satisfying $\min(L) \geq 1/\delta$, (71) and (38) yield that for all $B \in \text{MAX}(\mathcal{S}_\alpha) \cap [L]^{<\omega}$

$$(74) \quad \begin{aligned} \left\| \sum_{A \preceq B} \zeta(\alpha, A) x_A \right\| &\geq \left\| \sum_{A \preceq B} \zeta(\alpha, A') x_A \right\| - 2\delta \\ &= \left\| 0x_{\{\min(B)\}} + \sum_{\{\min(B)\} \prec A \preceq B} \zeta(\alpha, A') x_{A'}^B \right\| - 2\delta \\ &= \left\| \sum_{A \prec B} \zeta(\alpha, A) x_A^B \right\| - 2\delta \geq \varepsilon - 3\delta. \end{aligned}$$

For $B \in \text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ and $i = 0, 1, 2$ define

$$B^{(i)} = \{A \preceq B : \#A \bmod 3 = i\}.$$

By the triangle inequality, for some $0 \leq i(B) \leq 2$, we have $\|\sum_{A \in B^{(i(B))}} \zeta(\alpha, A) x_A\| \geq \varepsilon/3 - \delta$. By Proposition 2.14, we may pass to some infinite subset of L , again denoted L , so that for all $B \in \text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$, we have $i(B) = i_0$, for some common $i_0 \in \{0, 1, 2\}$. We shall assume that $i_0 = 0$, as the other cases are treated similarly. Therefore, for all $B \in \text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ we have

$$(75) \quad \left\| \sum_{A \in B^{(0)}} \zeta(\alpha, A) x_A \right\| \geq \frac{\varepsilon}{3} - \delta.$$

Lemma 3.7 parts (iii) and (v) also imply the following. If $B \in \text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$, then

$$(76) \quad \sum_{\substack{A \in B^{(0)}: \\ l_1(A')=0}} \zeta(\alpha, A) + \sum_{\substack{A \in B^{(0)}: \\ l_1(A'')=0}} \zeta(\alpha, A) + \sum_{\substack{A \in B^{(0)}: \\ l_1(A''')=0}} \zeta(\alpha, A) \leq \frac{3}{\min(B)} \leq 3\delta.$$

Hence, if for $B \in \text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ we set

$$\hat{B}^{(0)} = B^{(0)} \setminus \{A \preceq B : l_1(A') = 0, \text{ or } l_1(A'') = 0, \text{ or } l_1(A''') = 0\},$$

then

$$(77) \quad \left\| \sum_{A \in \hat{B}^{(0)}} \zeta(\alpha, A) x_A \right\| \geq \frac{\varepsilon}{3} - 4\delta.$$

If $L = \{\ell_1, \ell_2, \dots, \ell_k, \dots\}$, define $N = \{\ell_3, \ell_6, \dots, \ell_{3k}, \ell_{3(k+1)}, \dots\}$.

For each $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$, with $(\#A) \bmod 3 = 0$, we define $\tilde{A} \in [L]^{<\omega}$ as described bellow. If $A = \{a_1, \dots, a_d\}$, where $a_j = \ell_{3b_j}$ and $A_j = \{a_1, \dots, a_j\}$ for $1 \leq j \leq d$, we define the elements of a set $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ in groups of three as follows. If $j \bmod 3 = 0$ we put

$$(\tilde{a}_{j-2}, \tilde{a}_{j-1}, \tilde{a}_j) = \begin{cases} (a_{j-2}, a_{j-1}, a_j) & \text{if } l_1(A'_j) = 0 \text{ or } l_1(A'_{j-1}) = 0 \text{ or } l_1(A'_{j-2}) = 0 \\ (\ell_{3b_j-2}\ell_{3b_j-1}, a_j) & \text{if } l_1(A'_j), l_1(A'_{j-1}), l_1(A'_{j-2}) \neq 0. \end{cases}$$

It is not hard to see that A and \tilde{A} satisfy the assumptions of Lemma 3.8, hence $\tilde{A} \in \mathcal{S}_\alpha \cap [L]^{<\omega}$ and $\zeta(\alpha, A) = \zeta(\alpha, \tilde{A})$. Observe the following:

- (a) If $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ and $A^{(1)} \prec A^{(2)}$ are in $\hat{B}^{(0)}$, then $\tilde{A}^{(1)} \prec \tilde{A}^{(2)}$.
- (b) If $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ and $A \in \hat{B}^{(0)}$,

$$(78) \quad \text{if } \max(A) = \ell_{3n} \text{ then we have } \max(\tilde{A}) = \ell_{3n-2}.$$

(a) is clear, while (b) follows from the fact that $A \in \hat{B}^{(0)}$, implies that d is divisible by 3 and $l_1(A'') \neq 0$.

We shall define a weakly null tree $(z_A)_{A \in \mathcal{S}_\alpha \cap [N]^{<\omega}}$ so that for all $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^\infty)$ we have

$$(79) \quad \left\| \sum_{A \preceq B} \zeta(\alpha, A) z_A \right\| \geq \frac{\varepsilon}{3} - 4\delta.$$

The choice of δ and statement (iv) of Theorem 5.3 will yield a contradiction.

For $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ define

$$z_A = \begin{cases} x_{\tilde{A}} & \text{if } \#A \bmod 3 = 0 \text{ and } l_1(A'), l_1(A''), l_1(A''') \neq 0, \\ 0 & \text{else.} \end{cases}$$

Let $C \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$ and, by (a) we can find B be in $\text{MAX}(\mathcal{S}_\alpha \cap [L]^{<\omega})$ so that $\tilde{A} \preceq B$, for all $A \preceq C$, with $\#A = 0 \bmod 3$. Then, one can verify that

$$\left\| \sum_{A \preceq C} \zeta(\alpha, A) z_A \right\| = \left\| \sum_{A \preceq C} \zeta(\alpha, \tilde{A}) x_{\tilde{A}} \right\| = \left\| \sum_{A \in \hat{C}^{(0)}} \zeta(\alpha, \tilde{A}) x_{\tilde{A}} \right\| = \left\| \sum_{A \in \hat{B}^{(0)}} \zeta(\alpha, A) x_A \right\| \geq \frac{\varepsilon}{3} - 4\delta.$$

Let now $A \in \mathcal{S}_\alpha \cap [N]^{<\omega}$ be non-maximal. We will show that $\text{w-lim}_{n \in N} z_{A \cup \{n\}} = 0$. By the definition of the vectors z_A , we need only treat the case in which $(\#A + 1) \bmod 3 = 0$, i.e. when $z_{A \cup \{n\}} = x_{\widetilde{A \cup \{n\}}}$ for all $n \in N$ with $n > \max(A)$. In this case, by (73), we deduce that if $\ell_{3n} \in N$, then

$$\min \text{supp}(z_{A \cup \{\ell_{3n}\}}) = \min \text{supp}(x_{\widetilde{A \cup \{\ell_{3n}\}}}) > \max((A \cup \{\ell_{3n}\})'') = \ell_{3n-2}$$

where the last equality follows by (b). Hence, $\lim_{n \in N} \min \text{supp}(z_{A \cup \{n\}}) = \infty$. The fact that the FDD of X is shrinking completes the proof. \square

7. TWO METRICS ON \mathcal{S}_α , $\alpha < \omega_1$

Since $[N]^{<\omega}$ with \prec is a tree, with a unique root \emptyset , we could consider on $[N]^{<\omega}$ the usual *tree distance* which we denote by d : For $A = \{a_1, a_2, \dots, a_l\} \in [N]^{<\omega}$, or $B = \{b_1, b_2, \dots, b_m\}$, we let $n = \max\{j \geq 0 : a_i = b_i \text{ for } i = 1, 2, \dots, j\}$, and then let $d(A, B) = l + m - 2j$. But this distance will not lead to the results we are seeking. Indeed, it was shown in [9, Theorem 1.2] that for any reflexive space X $[N]^{<\omega}$, with the graph metric embeds bi-Lipschitzly into X iff and only if $\max(\text{Sz}(X), \text{Sz}(X^*)) > \omega$. We will need *weighted graph metrics* on \mathcal{S}_α .

- (i) The *weighted tree distance on \mathcal{S}_α* . For A, B in \mathcal{S}_α let C be the largest element in \mathcal{S}_α (with respect to \prec) so that $C \preceq A$ and $C \preceq B$ (i.e., C is the common initial segment of A and B) and then let

$$d_{1,\alpha}(A, B) = \sum_{a \in A \setminus C} z_{(\alpha, A)}(a) + \sum_{b \in B \setminus C} z_{(\alpha, B)}(b) = \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D).$$

- (ii) The *weighted interlacing distance on \mathcal{S}_α* can be defined as follows. For $A, B \in \mathcal{S}_\alpha$, say $A = \{a_1, a_2, \dots, a_l\}$ and $B = \{b_1, b_2, \dots, b_m\}$, with $a_1 < a_2 < \dots < a_l$ and $b_1 < b_2 < \dots < b_m$, we put $a_0 = b_0 = 0$ and $a_{l+1} = b_{m+1} = \infty$, and define

$$d_{\infty, \alpha}(A, B) = \max_{i=1, \dots, m+1} \sum_{a \in A, b_{i-1} < a < b_i} z_{(\alpha, A)}(a) + \max_{i=1, \dots, l+1} \sum_{b \in B, a_{i-1} < b < a_i} z_{(\alpha, B)}(b).$$

Remark. In order to explain $d_{\infty, \alpha}$ let us take some sets $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$ in \mathcal{S}_α and fix some $i \in \{0, 1, 2, \dots, m\}$. Now we measure how large the part of B is which lies between a_i and a_{i+1} (as before $a_0 = 0$ and $a_{m+1} = \infty$) by putting

$$m_i(B) := \sum_{j, b_j \in (a_i, a_{i+1})} \zeta(\alpha, \{b_1, b_2, \dots, b_j\}).$$

Then we define $m_j(A)$, for $j = 1, 2, \dots, n$ similarly, and put

$$d_{\infty, \alpha}(A, B) = \max_{1 \leq i \leq m} m_i(B) + \max_{1 \leq j \leq n} m_j(A).$$

We note that if C is maximal so that $C \preceq A$ and $C \preceq B$, and if $A \setminus C < B \setminus C$, then $d_{1,\alpha}(A, B) = d_{\infty, \alpha}(A, B)$.

Proposition 7.1. *The metric space $(\mathcal{S}_\alpha, d_{1,\alpha})$ is stable, i.e., for any sequences (A_n) and (B_n) in \mathcal{S}_α and any ultrafilter \mathcal{U} on \mathbb{N} it follows that*

$$\lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} d_{1,\alpha}(A_m, B_n) = \lim_{n \in \mathcal{U}} \lim_{m \in \mathcal{U}} d_{1,\alpha}(A_m, B_n),$$

while $(\mathcal{S}_\alpha, d_{\infty, \alpha})$ is not stable.

Proof. Since \mathcal{S}_α is compact in the product topology we can pass to subsequences and assume that we can write A_n as $A_n = A \cup \tilde{A}_n$, with $\max(A) < \min(\tilde{A}_n) \leq \max(\tilde{A}_n) < \min(\tilde{A}_{n+1})$, and B_n as $B_n = A \cup \tilde{B}_n$, with $\max(B) < \min(\tilde{B}_n) \leq \max \tilde{B}_n < \min \tilde{B}_{n+1}$, for $n \in \mathbb{N}$. We can also assume the sequences

$$(r_n^A) = \left(\sum_{A \prec D \preceq \tilde{A}_n} \zeta(\alpha, D) \right) \text{ and } (r_n^B) = \left(\sum_{B \prec D \preceq \tilde{B}_n} \zeta(\alpha, D) \right)$$

converge to some numbers r^A and r^B respectively.

Let $C \in \mathcal{S}_\alpha$ be the maximal element for which $C \preceq A$ and $C \preceq B$. For $m \in \mathbb{N}$ large enough and $n > m$, large enough (depending on m) it follows that

$$d_{1,\alpha}(A_m, B_n) = \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) + r_m^A + r_n^B.$$

Also for $n \in \mathbb{N}$ large enough and $m > n$, large enough (depending on n) it follows also that

$$d_{1,\alpha}(A_m, B_n) = \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) + r_m^A + r_n^B.$$

Thus we have

$$\begin{aligned} \lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} d_{1,\alpha}(A_m, B_n) &= \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) + r^A + r^B \\ &= \lim_{n \in \mathcal{U}} \lim_{m \in \mathcal{U}} d_{1,\alpha}(A_m, B_n). \end{aligned}$$

To prove our second claim we let (A_n) be a sequence of nonempty elements in \mathcal{S}_α with $d(\emptyset, A_n) = \sum_{D \preceq A_n} \zeta(\alpha, D) < 1/3$, and $A_{n-1} < \min A_n$, for $n \in \mathbb{N}$. Then we let $B \in \mathcal{S}_\alpha$ with $d(\emptyset, B) = \sum_{D \preceq B} \zeta(\alpha, D) \in (1/3, 1/2]$, and then choose $\tilde{B}_n \in \mathcal{S}_\alpha$ so that $\max B < \min \tilde{B}_n \leq \max \tilde{B}_n < \min \tilde{B}_{n-1}$, and so that $B_n = B \cup B'_n$ is maximal in \mathcal{S}_α , for $n \in \mathbb{N}$.

Then it follows

$$\lim_{m \in \mathcal{U}} \lim_{n \in \mathcal{U}} d_{\infty,\alpha}(A_m, B_n) = 1 - d(\emptyset, B) \leq 2/3$$

and

$$\lim_{n \in \mathcal{U}} \lim_{m \in \mathcal{U}} d_{\infty,\alpha}(A_m, B_n) \geq 1.$$

□

We can now conclude one direction of Theorem A from James' characterization of reflexive spaces.

Proposition 7.2. *If X is a non reflexive Banach space then for any $0 < c < 1/4$ and every $\alpha > 0$ there is a map $\Phi_\alpha : \mathcal{S}_\alpha \rightarrow X$, so that*

$$(80) \quad cd_{\infty,\alpha}(A, B) \leq \|\Phi(A) - \Phi(B)\| \leq d_{1,\alpha}(A, B) \text{ for all } A, B \in \mathcal{S}_\alpha.$$

Remark. Our argument will show that if X is non reflexive there is a sequence $(x_n) \subset B_X$ so that for all $\alpha < \omega_1$ the map

$$\Phi_\alpha : \mathcal{S}_\alpha \rightarrow X, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) x_{\max(D)},$$

satisfies (80).

Proof. Let θ be any number in $(0, 1)$. Then by [16] there is a normalized basic sequence in X whose basic constant is at most $\frac{2}{\theta}$ satisfying

$$(81) \quad \left\| \sum_{j=1}^{\infty} a_j x_j \right\| \geq \theta \sum_{j=1}^{\infty} a_j \text{ for all } (a_j) \in c_{00}, a_j \geq 0, \text{ for all } j \in \mathbb{N}.$$

Thus its bimonotonicity constant is at most $\frac{4}{\theta}$, which means that for $m \leq n$ the projection

$$P_{[m,n]} : \overline{\text{span}(x_j)} \rightarrow \overline{\text{span}(x_j)}, \quad \sum_{j=1}^{\infty} a_j x_j \mapsto \sum_{j=m}^n a_j x_j,$$

has norm at most $\frac{4}{\theta}$.

We define

$$\Phi : \mathcal{S}_\alpha \rightarrow X, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) x_{\max(D)}.$$

For $A, B \in \mathcal{S}_\alpha$ we let C be the maximal element in \mathcal{S}_α for which $C \preceq A$ and $C \preceq B$. Then

$$\|\Phi(A) - \Phi(B)\| = \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) x_{\max D} - \sum_{C \prec D \preceq B} \zeta(\alpha, D) x_{\max(D)} \right\|$$

$$\leq \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B).$$

On the other hand, if we write $A = \{a_1, a_2, \dots, a_l\}$ and put $a_0 = 0$, and $a_{l+1} = \infty$, it follows for all $i = 1, 2, \dots, l+1$, that

$$\|\Phi(A) - \Phi(B)\| \geq \frac{\Theta}{4} \|P_{(a_{i-1}, a_i)}(\Phi(A) - \Phi(B))\| \geq \frac{\Theta^2}{4} \sum_{a_{i-1} < b < a_i} z_{(\alpha, B)}(b).$$

Similarly, if we write $B = \{b_1, b_2, \dots, b_m\}$ and put $b_0 = 0$, and $b_{m+1} = \infty$, it follows for all $j = 1, 2, \dots, m+1$ that

$$\|\Phi(A) - \Phi(B)\| \geq \frac{\Theta^2}{4} \sum_{b_{j-1} < a < b_j} z_{(\alpha, B)}(a).$$

Thus for any $i = 1, 2, \dots, l$ and any $j = 1, 2, \dots, m$

$$\|\Phi(A) - \Phi(B)\| \geq \frac{\Theta^2}{8} \left[\sum_{a_{i-1} < b < a_i} z_{(\alpha, B)}(b) + \sum_{b_{j-1} < a < b_j} z_{(\alpha, B)}(a) \right],$$

which implies our claim. \square

We finish this section with an observation which we will need later.

Lemma 7.3. *Let ξ and γ be countable ordinal numbers with $\gamma < \beta = \omega^\xi$. Let $B_1 < \dots < B_j$ be in $\text{MAX}(\mathcal{S}_{\beta\gamma})$, so that $\bar{B} = \{\min(B_j) : 1 \leq j \leq d\}$ is a non-maximal \mathcal{S}_β set with $l_1(\bar{B}) > 0$ ($l_1(A)$ for $A \in [\mathbb{N}]^{<\omega}$ has been defined before Lemma 3.7) and set $D = \cup_{j=1}^d B_j \in \mathcal{S}_{\beta(\gamma+1)}$. Then for every A, B in $\mathcal{S}_{\beta\gamma}$ with $D < A$ and $D < B$ we have*

$$(82) \quad d_{1,\beta\gamma}(A, B) = \frac{1}{\zeta(\beta, \bar{B})} d_{1,\beta(\gamma+1)}(D \cup A, D \cup B) \text{ and}$$

$$(83) \quad d_{\infty,\beta\gamma}(A, B) = \frac{1}{\zeta(\beta, \bar{B})} d_{\infty,\beta(\gamma+1)}(D \cup A, D \cup B).$$

Proof. We will only prove (82), as the proof of (83) uses the same argument. Let C be the maximal element in $\mathcal{S}_{\beta\gamma}$ so that $C \preceq A$ and $C \preceq B$. Note that $\tilde{C} = D \cup C$ is the largest element of $\mathcal{S}_{\beta(\gamma+1)}$ so that $\tilde{C} \preceq D \cup A$ and $\tilde{C} \preceq D \cup B$. Define $\tilde{B}_1 = \bar{B} \cup \{\min(A)\}$ and $\tilde{B}_2 = \bar{B} \cup \{\min(B)\}$ and observe that, since $l_1(\bar{B}) > 0$, $\zeta(\beta, \tilde{B}_1) = \zeta(\beta, \tilde{B}_2) = \zeta(\beta, \bar{B})$. Using (34) in Proposition 3.4 we conclude the following:

$$(84) \quad \sum_{a \in (D \cup A) \setminus \tilde{C}} z_{(\beta(\gamma+1), D \cup A)}(a) = \zeta(\beta, \tilde{B}_1) \sum_{a \in A \setminus C} z_{(\beta\gamma, A)}(a) = \zeta(\beta, \bar{B}) \sum_{a \in A \setminus C} z_{(\beta\gamma, A)}(a)$$

and similarly we obtain

$$(85) \quad \sum_{a \in (D \cup B) \setminus \tilde{C}} z_{(\beta(\gamma+1), D \cup B)}(a) = \zeta(\beta, \bar{B}) \sum_{a \in B \setminus C} z_{(\beta\gamma, B)}(a).$$

Applying (84) and (85) to the definition of the $d_{1,\alpha}$ metrics, the result easily follows. \square

8. THE SZLENK INDEX AND EMBEDDINGS OF $(\mathcal{S}_\alpha, d_{1,\alpha})$ INTO X

In this section we show the following two results, Theorem 8.1 and 8.3, which establish a proof of Theorem B.

Theorem 8.1. *Let X be a separable Banach space and α a countable ordinal. Assume that $\text{Sz}(X) > \omega^\alpha$ then $(\mathcal{S}_\alpha, d_{1,\alpha})$ can be bi-Lipschitzly embedded into X and X^* .*

Before proving Theorem 8.1 we want to first cover the case that ℓ_1 embeds into X .

Example 8.2. For each $\alpha < \omega_1$ we want to define a bi-Lipschitz embedding of $(\mathcal{S}_\alpha, d_{1,\alpha})$ into a Banach space X and its dual X^* under the assumption that ℓ_1 embeds into X . We first choose for every $A \in [\mathbb{N}]^{<\omega}$ a spread \tilde{A} of A as in Example 5.6. Then we define for $\alpha < \omega_1$

$$\Phi : \mathcal{S}_\alpha \rightarrow \ell_1, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) e_{\max(\tilde{D})}.$$

Since for $A, B \in \mathcal{S}_\alpha$ it follows that

$$\begin{aligned} \|\Phi(A) - \Phi(B)\| &= \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) e_{\max(\tilde{D})} - \sum_{C \prec D \preceq B} \zeta(\alpha, D) e_{\max(\tilde{D})} \right\| \\ &= \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B), \end{aligned}$$

where $C \in \mathcal{S}_\alpha$ is the maximal element for which $C \preceq A$ and $C \preceq B$, it follows that Φ is an isometric embedding of $(\mathcal{S}_\alpha, d_{1,\alpha})$ into ℓ_1 .

Thus, if ℓ_1 embeds into X then $(\mathcal{S}_\alpha, d_{1,\alpha})$ bi-Lipschitzly embeds into X . Secondly ℓ_∞ is a quotient of X^* in that case, and since ℓ_1 embeds into ℓ_∞ , it follows easily that ℓ_1 embeds into X^* , and, thus, that $(\mathcal{S}_\alpha, d_{1,\alpha})$ also bi-Lipschitzly embeds into X^* .

Proof of Theorem 8.1. Because of Example 8.2 we can assume that ℓ_1 does not embed into X . Thus, we can apply Theorem 5.3, (i) \iff (iii) and obtain $\varepsilon > 0$, a tree $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_{X^*}$ and a w -null tree $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, so that

$$(86) \quad z_B^*(z_A) > \varepsilon, \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \preceq B,$$

$$(87) \quad |z_B^*(z_A)| \leq \frac{\varepsilon}{2} \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \not\preceq B,$$

Then we define

$$\Phi : \mathcal{S}_\alpha \rightarrow X, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) z_D.$$

If $A, B \in \mathcal{S}_\alpha$ and $C \in \mathcal{S}_\alpha$ is the maximal element of \mathcal{S}_α for which $C \preceq A$ and $C \preceq B$, we note that

$$\begin{aligned} \|\Phi(A) - \Phi(B)\| &= \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \right\| \\ &\leq \sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) = d_{1,\alpha}(A, B). \end{aligned}$$

Moreover we obtain

$$\|\Phi(A) - \Phi(B)\| = \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \right\|$$

$$\begin{aligned}
&\geq z_A^* \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \right) \\
&\geq \varepsilon \sum_{C \prec D \preceq A} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq B} \zeta(\alpha, D).
\end{aligned}$$

Similarly we can show that

$$\|\Phi(A) - \Phi(B)\| \geq \varepsilon \sum_{C \prec D \preceq B} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq A} \zeta(\alpha, D),$$

and thus

$$\|\Phi(A) - \Phi(B)\| \geq \frac{\varepsilon}{4} \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) \right) = \frac{\varepsilon}{4} d_{1,\alpha}(A, B).$$

In order to define a Lipschitz embedding from $(\mathcal{S}_\alpha, d_{1,\alpha})$ into X^* , we also can assume that ℓ_1 does not embed into X . Because otherwise ℓ_∞ is a quotient of X^* , and thus since ℓ_{l_1} embeds into ℓ_∞ , it is easy to see that ℓ_1 embeds into X^* , and again we let

$$\Psi : \mathcal{S}_\alpha \rightarrow X^*, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) z_D^*.$$

As in the case of Φ it is easy to see that Ψ is a Lipschitz function with constant not exceeding the value 1. Again if $A, B \in \mathcal{S}_\alpha$ let $C \in \mathcal{S}_\alpha$ be the maximal element of \mathcal{S}_α for which $C \preceq A$ and $C \preceq B$. In the case that $C \prec A$, we let $C^+ \in \mathcal{S}_\alpha$ be the minimal element for which $C \prec C^+ \preceq A$. We note that $C^+ \not\preceq D$ for any $D \in \mathcal{S}_\alpha$ with $C \prec D \preceq B$, and it follows therefore that

$$\begin{aligned}
\|\Psi(A) - \Psi(B)\| &= \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D^* - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D^* \right\| \\
&\geq \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D^* - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D^* \right) (z_{C^+}) \\
&\geq \varepsilon \sum_{C \prec D \preceq A} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq B} \zeta(\alpha, D).
\end{aligned}$$

If $C = A$ we arrive trivially to the same inequality. Similarly we obtain that

$$\|\Psi(A) - \Psi(B)\| \geq \varepsilon \sum_{C \prec D \preceq B} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq A} \zeta(\alpha, D).$$

This yields

$$\|\Psi(A) - \Psi(B)\| \geq \frac{\varepsilon}{4} \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) \right) = \frac{\varepsilon}{4} d_{1,\alpha}(A, B)$$

which finishes the proof of our claim. \square

Theorem 8.3. *Assume that X is a Banach space having a separable dual X^* , with $\text{Sz}(X^*) > \omega^\alpha$ then $(\mathcal{S}_\alpha, d_{1,\alpha})$ can be bi-Lipschitzly embedded into X .*

Proof. Applying Proposition 5.5 we obtain $\varepsilon > 0$ and trees $(z_A^* : A \in \mathcal{S}_\alpha) \subset B_X^*$ and $(z_A : A \in \mathcal{S}_\alpha) \subset B_X$, so that $(z_A^* : A \in \mathcal{S}_\alpha)$ is w^* -null and

$$(88) \quad z_A^*(z_B) > \varepsilon, \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \preceq B,$$

$$(89) \quad |z_A^*(z_B)| \leq \frac{\varepsilon}{2}, \text{ for all } A, B \in \mathcal{S}_\alpha \setminus \{\emptyset\}, \text{ with } A \not\preceq B,$$

We define

$$\Phi : \mathcal{S}_\alpha \rightarrow X, \quad A \mapsto \sum_{D \preceq A} \zeta(\alpha, D) z_A.$$

Again it is easy to see that Φ is a Lipschitz function with constant not exceeding the value 1. Moreover if $A, B \in \mathcal{S}_\alpha$ let $C \in \mathcal{S}_\alpha$ be the maximal element of \mathcal{S}_α for which $C \preceq A$ and $C \preceq B$. In the case that $C \prec A$, we let $C^+ \in \mathcal{S}_\alpha$ be the minimal element for which $C \prec C^+ \preceq A$. We note that $C^+ \not\preceq D$ for any $D \in \mathcal{S}_\alpha$ with $C \prec D \preceq B$, and it follows therefore that

$$\begin{aligned} \|\Phi(A) - \Phi(B)\| &= \left\| \sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \right\| \\ &\geq z_{C^+}^* \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) z_D - \sum_{C \prec D \preceq B} \zeta(\alpha, D) z_D \right) \\ &\geq \varepsilon \sum_{C \prec D \preceq A} \zeta(\alpha, D) - \frac{\varepsilon}{2} \sum_{C \prec D \preceq B} \zeta(\alpha, D) \eta(D). \end{aligned}$$

If $C = A$ we trivially deduce that

$$\|\Phi(A) - \Phi(B)\| \geq \varepsilon \sum_{C \prec D \preceq A} \zeta(\alpha, D) - \sum_{C \prec D \preceq B} \zeta(\alpha, D) \eta(D).$$

Similarly we prove that

$$\|\Phi(A) - \Phi(B)\| \geq \varepsilon \sum_{C \prec D \preceq B} \zeta(\alpha, D) - \sum_{C \prec D \preceq A} \zeta(\alpha, D) \eta(D).$$

We obtain therefore that

$$\begin{aligned} \|\Phi(A) - \Phi(B)\| &\geq \frac{1}{2} \left(\sum_{C \prec D \preceq A} (\varepsilon - \eta(A)) \zeta(\alpha, D) + \sum_{C \prec D \preceq B} (\varepsilon - \eta(A)) \zeta(\alpha, D) \right) \\ &\geq \frac{\varepsilon}{4} \left(\sum_{C \prec D \preceq A} \zeta(\alpha, D) + \sum_{C \prec D \preceq B} \zeta(\alpha, D) \right) = \frac{\varepsilon}{4} d_{1,\alpha}(A, B) \end{aligned}$$

which proves our claim. \square

9. REFINEMENT ARGUMENT

Before providing a proof of Theorem C, and, thus, the still missing implication of Theorem A, we will introduce in this and the next section some more notation and make some preliminary observations. We will consider maps $\Phi : \mathcal{S}_\alpha \rightarrow X$, satisfying weaker conditions compared to the ones required by Theorem A and C. On the one hand it will make an argument using transfinite induction possible, on the other hand it is sufficient to arrive to the wanted conclusions.

Definition 9.1. Let $\alpha < \omega_1$. For $r \in (0, 1]$ we define

$$\mathcal{S}_\alpha^{(r)} = \left\{ A \in \mathcal{S}_\alpha : \sum_{D \preceq A} \zeta(\alpha, D) \leq r \right\}.$$

It is not hard to see that $\mathcal{S}_\alpha^{(r)}$ is a closed subset of \mathcal{S}_α , and hence compact, and closed under restrictions. We also put

$$\mathcal{M}_\alpha^{(r)} = \text{MAX}(\mathcal{S}_\alpha^{(r)}) = \{A \in \mathcal{S}_\alpha^{(r)} : A \text{ is maximal in } \mathcal{S}_\alpha^{(r)} \text{ with resp. to } \prec\}$$

and for $A \in \mathcal{S}_\alpha$,

$$\mathcal{M}_\alpha^{(r)}(A) = \{B \in \mathcal{S}_\alpha(A) : A \cup B \in \mathcal{M}_\alpha^{(r)}\}.$$

Definition 9.2. Let X be a Banach space, α be a countable ordinal number, L be an infinite subset of \mathbb{N} , and A_0 be a set in \mathcal{S}_α that is either empty or a singleton. A map $\Phi : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \rightarrow X$ is called a *semi-embedding of $\mathcal{S}_\alpha \cap [L]^{<\omega}$ into X starting at A_0* , if there is a number $c > 0$ so that

$$(90) \quad \|\Phi(A) - \Phi(B)\| \leq d_{1,\alpha}(A_0 \cup A, A_0 \cup B) \text{ for all } A, B \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}, \text{ and}$$

for all $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$, with $l_1(A_0 \cup A) > 0$, for all $r \in (0, 1]$, and for all B_1, B_2 in $\mathcal{M}^{(r)}(A_0 \cup A) \cap [L]^{<\omega}$ with $B_1 < B_2$:

$$(91) \quad \|\Phi(A \cup B_1) - \Phi(A \cup B_2)\| \geq cd_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2).$$

($l_1(A)$ for $A \in [\mathbb{N}]^{<\omega}$ was introduced in Definition 3.6).

We call the supremum of all numbers $c > 0$ so that (91) holds for all $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$ and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [L]^{<\omega}$, with $B_1 < B_2$, the *semi-embedding constant of Φ* and denote it by $c(\Phi)$.

Remark. If $\Phi : \mathcal{S}_\alpha \rightarrow X$ is for some $0 < c < C$ a c -lower $d_{\infty,\alpha}$ and C -upper $d_{1,\alpha}$ embedding, we can, after rescaling Φ if necessary, assume that $C = 1$, and from the definition of $d_{1,\alpha}$ and $d_{\infty,\alpha}$ we can easily see that for every A_0 , that is either empty or a singleton, and L in $[\mathbb{N}]^\omega$, the restrictions $\Phi|_{\mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}} : X$ is a semi-embedding.

Assume $\Phi : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \rightarrow X$, is a semi-embedding of $\mathcal{S}_\alpha \cap [L]^{<\omega}$ into X starting at A_0 . For $A \in \mathcal{S}_\alpha(A_0)$, with $A \neq \emptyset$, we put $A' = A \setminus \{\max(A)\}$ and define

$$x_{A_0 \cup A} = \frac{1}{\zeta(\alpha, A_0 \cup A)} (\Phi(A) - \Phi(A')).$$

If $A_0 = \emptyset$ put $x_\emptyset = \Phi(\emptyset)$, whereas if A_0 is a singleton, define $x_\emptyset = 0$ and $x_{A_0} = (1/\zeta(\alpha, A_0))\Phi(\emptyset)$. Note that $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}\} \subset B_X$. Recall that $\zeta(\alpha, \emptyset) = 0$ and hence, for $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$, we have

$$\Phi(A) = x_\emptyset + \sum_{\emptyset \preceq D \preceq A_0 \cup A} \zeta(\alpha, D)x_D.$$

We say in that case that *the family $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}\}$ generates Φ* .

In that case the map $\Phi_0 : \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega} \rightarrow X$ with $\Phi_0 = \Phi - x_\emptyset$, i.e. for $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$

$$(92) \quad \Phi_0(A) = \sum_{\emptyset \preceq D \preceq A_0 \cup A} \zeta(\alpha, D)x_D,$$

is also a semi-embedding of $\mathcal{S}_\alpha \cap [L]^{<\omega}$ into X starting at A_0 , with $c(\Phi_0) = c(\Phi)$.

Lemma 9.3. *Let $\gamma, \xi < \omega_1$, with $\gamma < \beta = \omega^\xi$, and let $B_1 < \dots < B_d$ be in $\text{MAX}(\mathcal{S}_{\beta\gamma})$, so that $\bar{B} = \{\min(B_j) : 1 \leq j \leq d\}$ is a non-maximal \mathcal{S}_β set with $l_1(\bar{B}) > 0$. Set $D = \cup_{j=1}^d B_j$, let $r \in (0, 1]$ and let also $A \in \mathcal{M}_{\beta\gamma}^{(r)}$ with $D < A$. Then, if $r_0 = \sum_{C \preceq D} \zeta(\beta(\gamma+1), C) + \zeta(\beta, \bar{B})r$, we have that $A \in \mathcal{M}_{\beta(\gamma+1)}^{(r_0)}(D)$.*

Proof. From Proposition 3.4 and Lemma 3.7 (iii) we obtain that for $C \preceq A$ we have

$$\zeta(\beta(\gamma+1), D \cup C) = \zeta(\beta, \bar{B} \cup \{\min(A)\})\zeta(\beta\gamma, C) = \zeta(\beta, \bar{B})\zeta(\beta\gamma, C),$$

which implies that $D \cup A \in \mathcal{S}_{\beta(\gamma+1)}^{(r_0)}$. If we assume that $D \cup A$ is not in $\mathcal{M}_{\beta(\gamma+1)}^{(r_0)}$, there is $B \in \mathcal{S}_{\beta(\gamma+1)}^{(r_0)}$ with $D \cup A \prec B$. Possibly after trimming B , we may assume that $B' = D \cup A$. Define $B_0 = B \setminus D$. Evidently, $A \prec B_0$ and $B'_0 = A$. We claim that $B_0 \in \mathcal{S}_{\beta\gamma}$. If we assume that this is not the case, A is a maximal $\mathcal{S}_{\beta\gamma}$ set. This yields that $\sum_{C \preceq A} \zeta(\beta\gamma, C) = 1$ and hence $r = 1$ and $r_0 = \sum_{C \preceq D} \zeta(\beta(\gamma+1), C) + \zeta(\beta, \bar{B}) = \sum_{C \preceq D \cup A} \zeta(\beta(\gamma+1), C)$, i.e. $D \cup A \in \mathcal{M}_{\beta(\gamma+1)}^{(r_0)}$ which we assumed to be false. Thus, we conclude that $B_0 \in \mathcal{S}_{\beta\gamma}$ and thus, using Proposition 3.4 and the definition of r_0 , $B_0 \in \mathcal{S}_{\beta\gamma}^{(r)}$, which is a contradiction, as $A \in \mathcal{M}_{\beta\gamma}^{(r)}$ and $A \prec B_0$. \square

Lemma 9.4. *Let $\alpha < \omega_1$, $N \in [\mathbb{N}]^{<\omega}$, A_0 be a subset of \mathbb{N} that is either empty or a singleton, and $c \in (0, 1]$. Let $\Psi : \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_\alpha \cap [N]^{<\omega}$ into X starting at A_0 , generated by a family of vectors $\{z_\emptyset\} \cup \{z_{A \cup A_0} : A \in \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}\}$, so that $c(\Psi) > c$. Let $\varepsilon < c(\Psi) - c$ and $\{\tilde{z}_\emptyset\} \cup \{\tilde{z}_{A \cup A_0} : A \in \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}\} \subset B_X$, with $\|\tilde{z}_{A_0 \cup A} - z_{A_0 \cup A}\| < \varepsilon$, for all $A \in \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}$ with $A_0 \cup A \neq \emptyset$. Then the map $\tilde{\Psi} : \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega} \rightarrow X$ defined by*

$$\tilde{\Psi}(A) = \sum_{D \preceq A_0 \cup A} \zeta(\alpha, D) \tilde{z}_D, \text{ for } A \in \mathcal{S}_\alpha \cap [N]^{<\omega},$$

is a semi-embedding of $\mathcal{S}_\alpha \cap [N]^{<\omega}$ into X starting at A_0 with $c(\tilde{\Psi}) > c$.

Proof. For any $r \in (0, 1]$, any $A \in \mathcal{S}_\alpha(A_0) \cap [L]^{<\omega}$ and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [N]^{<\omega}$, with $B_1 < B_2$ we obtain

$$\begin{aligned} \left\| \tilde{\Psi}(A \cup B_1) - \tilde{\Psi}(A \cup B_2) \right\| &= \left\| \sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_1} \zeta(\alpha, D) \tilde{z}_D - \sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_2} \zeta(\alpha, D) \tilde{z}_D \right\| \\ &\geq \left\| \sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_1} \zeta(\alpha, D) z_D - \sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_2} \zeta(\alpha, D) z_D \right\| \\ &\quad - \varepsilon \left(\sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_1} \zeta(\alpha, D) + \sum_{A_0 \cup A \prec D \preceq A_0 \cup A \cup B_2} \zeta(\alpha, D) \right) \\ &\geq (c(\Psi) - \varepsilon) d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2), \end{aligned}$$

which implies (91). (90) follows from the fact that $\tilde{z}_{A_0 \cup A} \in B_X$ for all $A \in \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}$ with $A_0 \cup A \neq \emptyset$. \square

For the rest of the section we will assume that X has a bimonotone FDD (E_n) . For finite or cofinite sets $A \subset \mathbb{N}$, we denote the canonical projections from X onto $\text{span}(E_j : j \in A)$

by P_A , i.e.,

$$P_A : X \rightarrow X, \quad \sum_{j=1}^{\infty} x_j \mapsto \sum_{j \in A} x_j, \quad \text{for } x = \sum_{j=1}^{\infty} x_j \in X, \text{ with } x_j \in E_j, \text{ for } j \in \mathbb{N},$$

and we write P_j instead of $P_{\{j\}}$, for $j \in \mathbb{N}$. We denote the linear span of the E_j by $c_{00}(E_j : j \in \mathbb{N})$, i.e.,

$$c_{00}(E_j : j \in \mathbb{N}) = \left\{ \sum_{j=1}^{\infty} x_j : x_j \in E_j, \text{ for } j \in \mathbb{N} \text{ and } \#\{j : x_j \neq 0\} < \infty \right\}.$$

Definition 9.5. Let α be a countable ordinal number, $M \in [\mathbb{N}]^{\omega}$, and A_0 be a subset of \mathbb{N} that is either empty or a singleton. A semi-embedding $\Phi : \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega} \rightarrow X$ of $\mathcal{S}_{\alpha} \cap [M]^{<\omega}$ into X starting at A_0 , is said to be c -refined, for some $c \leq c(\Phi)$, if the following conditions are satisfied:

- (i) the family $\{x_{\emptyset}\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega}\}$ generating Φ is contained in $B_X \cap c_{00}(E_j : j \in \mathbb{N})$,
- (ii) for all $A \in \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega}$ with $A_0 \cup A \neq \emptyset$ we have

$$\max(A_0 \cup A) \leq \max \text{supp}(x_{A_0 \cup A}) < \min\{m \in M : m > \max(A_0 \cup A)\}.$$

- (iii) for all $r \in (0, 1]$, $A \in \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega}$, with $l_1(A_0 \cup A) > 0$, and B_1, B_2 in $\mathcal{M}_{\alpha}^{(r)}(A_0 \cup A) \cap [M]^{<\omega}$, with $B_1 < B_2$, we have

$$\|P_{(\max \text{supp}(x_{A_0 \cup A}), \infty)}(\Phi(A \cup B_1) - \Phi(A \cup B_2))\| \geq cd_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2),$$

- (iv) for all $r \in (0, 1]$, $A \in \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega}$ and B in $\mathcal{M}_{\alpha}^{(r)}(A_0 \cup A) \cap [M]^{<\omega}$ we have

$$\|P_{(\max \text{supp}(x_{A_0 \cup A}), \infty)}(\Phi(A \cup B))\| \geq \frac{c}{2} \sum_{A_0 \cup A < C \preceq A_0 \cup A \cup B} \zeta(\alpha, C).$$

Remark 9.6. Let $\xi < \omega_1$, $\gamma \leq \beta = \omega^{\omega^{\xi}}$ be a limit ordinal, $0 < c \leq 1$, and $M \in [\mathbb{N}]^{\omega}$. If $a_0 \in \mathbb{N}$, we note that $\mathcal{S}_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega} = \mathcal{S}_{\beta\eta(\gamma, a_0)}(\{a_0\}) \cap [M]^{<\omega}$, and $\zeta(\beta\gamma, \{a_0\} \cup D) = \zeta(\beta\eta(\gamma, a_0), \{a_0\} \cup D)$ for $D \in \mathcal{S}_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega}$, where $(\eta(\gamma, n))_{n \in \mathbb{N}}$ is the sequence provided by Proposition 2.6. It follows that a semi-embedding of $\mathcal{S}_{\beta\gamma}(\{a_0\}) \cap [M]^{<\omega}$ into X starting at $\{a_0\}$ that is c -refined, is a c -refined semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, n)}(\{a_0\}) \cap [M]^{<\omega}$ into X .

Secondly, if $\Phi : \mathcal{S}_{\beta\gamma} \cap [M]^{<\omega} \rightarrow X$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X starting at \emptyset that is c -refined, then for every $a_0 \in M$ and $N = M \cap [a_0, \infty)$ the map $\Psi = \Phi|_{\mathcal{S}_{\beta\eta(\gamma, a_0)}(\{a_0\}) \cap [N]^{<\omega}}$, is a semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, a_0)} \cap [N]^{<\omega}$ into X starting at $\{a_0\}$, that is c -refined. Furthermore, Ψ is generated by the family $\{x_{A_0 \cup A} : A \in \mathcal{S}_{\beta\eta(\gamma, a_0)}(\{a_0\}) \cap [N]^{<\omega}\}$, where $\{x_A : A \in \mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}\}$ is the family generating Φ .

Lemma 9.7. Let $\xi, \gamma < \omega_1$, with $\gamma < \beta = \omega^{\omega^{\xi}}$, $M \in [\mathbb{N}]^{\omega}$, and A_0 be a subset of \mathbb{N} that is either empty or a singleton. Let also $\Phi : \mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [M]^{\infty}$ into X starting at A_0 , that is c -refined. The family generating Φ is denoted by $\{x_{\emptyset}\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_{\alpha}(A_0) \cap [M]^{<\omega}\}$. Extend the set A_0 to a set $A_0 \cup A_1$, $A_0 < A_1$, which can be written as $A_0 \cup A_1 = \bigcup_{j=1}^k B_j \in \mathcal{S}_{\beta(\gamma+1)} \cap [A_0 \cup M]^{<\omega}$, where $B_1 < \dots < B_k$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ and $\bar{B} = \{\min(B_j) : 1 \leq j \leq k\}$ is a non-maximal \mathcal{S}_{β} set with $l_1(\bar{B}) > 0$.

Then, for $N = M \cap (\max(A_0 \cup A_1), \infty)$ and $n_0 = \max \text{supp}(x_{A_0 \cup A_1})$, the map

$$\Psi : \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \rightarrow X, \quad A \mapsto \frac{1}{\zeta(\beta, \bar{B})} P_{(n_0, \infty)}(\Phi(A_1 \cup A))$$

is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [N]^\omega$ into X starting at \emptyset , that is c -refined. Moreover, Ψ is generated by the family $\{z_A : A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}\}$, where

$$(93) \quad z_\emptyset = 0 \text{ and } z_A = P_{(n_0, \infty)}(x_{A_0 \cup A_1 \cup A}) \text{ for } A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \setminus \{\emptyset\}.$$

Proof. By Lemma 7.3 we easily obtain that for $A, B \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$:

$$(94) \quad \|\Psi(A) - \Psi(B)\| \leq \frac{1}{\zeta(\beta, \bar{B})} d_{1, \beta(\gamma+1)}(A_0 \cup A_1 \cup A, A_0 \cup A_1 \cup B) = d_{1, (\beta\gamma)}(A, B),$$

i.e. (90) from Definition 9.2 is satisfied for Ψ . We will show that (90) from Definition 9.2 is satisfied for Ψ as well. Let $r \in (0, 1]$, A be in $\mathcal{S}_{\beta\gamma}$, with $l_1(A) > 0$, and $B_1 < B_2$ in $\mathcal{M}_{\beta\gamma}^{(r)}(A) \cap [N]^{<\omega}$ (i.e. $A \cup B_1, A \cup B_2 \in \mathcal{M}_{\beta\gamma}^{(r)}$). Note that we have $l_1(A_0 \cup A_1 \cup A) > 0$. If we set $r_0 = \sum_{C \preceq A_0 \cup A_1} \zeta(\beta(\gamma+1), C) + \zeta(\beta, \bar{B})r$, by Lemma 9.3 we deduce that $A \cup B_1$ and $A \cup B_2$ are in $\mathcal{M}_{\beta(\gamma+1)}^{(r_0)}(A_0 \cup A_1) \cap [N]^{<\omega}$, i.e., $B_1, B_2 \in \mathcal{M}_{\beta(\gamma+1)}^{(r_0)}(A_0 \cup (A_1 \cup A)) \cap [N]^{<\omega}$. Definition 9.5 (ii) implies $n_0 \leq \max \text{supp}(x_{A_0 \cup A_1 \cup A})$ and, thus, by Definition 9.5 (iii) for Φ we deduce

$$(95) \quad \begin{aligned} & \|\Psi(A \cup B_1) - \Psi(A \cup B_2)\| \\ &= \frac{1}{\zeta(\beta, \bar{B})} \|P_{(n_0, \infty)}(\Phi(A_1 \cup A \cup B_1) - \Phi(A_1 \cup A \cup B_2))\| \\ &\geq \frac{1}{\zeta(\beta, \bar{B})} \|P_{(\max \text{supp}(x_{A_0 \cup A_1 \cup A}), \infty)}(\Phi(A_1 \cup A \cup B_1) - \Phi(A_1 \cup A \cup B_2))\| \\ &\geq \frac{c}{\zeta(\beta, \bar{B})} d_{1, \beta(\gamma+1)}(A_0 \cup A_1 \cup A \cup B_1, A_0 \cup A_1 \cup A \cup B_2) \\ &= cd_{1, \beta\gamma}(A \cup B_1, A \cup B_2), \end{aligned}$$

where the last equality follows from Lemma 7.3. In particular, (94) and (95) yield that $\Psi : \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega} \rightarrow X$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [N]^\omega$ into X starting at \emptyset with $c(\Psi) \geq c$.

It remains to show that Ψ satisfies Definition 9.5 (i) to (iv). Observe that Definition 9.5 (ii) implies that for $C \preceq A_0 \cup A_1$, we have $P_{(n_0, \infty)}(x_C) = 0$. We combine this with (34) of Proposition 3.4 to obtain that for $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$:

$$(96) \quad \begin{aligned} \Psi(A) &= \frac{1}{\zeta(\beta, \bar{B})} P_{(n_0, \infty)}(\Phi(A_1 \cup A)) \\ &= \frac{1}{\zeta(\beta, \bar{B})} \sum_{C \preceq A_0 \cup A_1 \cup A} \zeta(\beta(\gamma+1), C) P_{(n_0, \infty)}(x_C) \\ &= \frac{1}{\zeta(\beta, \bar{B})} \sum_{A_0 \cup A_1 \prec C \preceq A_0 \cup A_1 \cup A} \zeta(\beta(\gamma+1), C) P_{(n_0, \infty)}(x_C) \\ &= \sum_{C \preceq A} \zeta(\beta\gamma, C) P_{(n_0, \infty)}(x_{A_0 \cup A_1 \cup C}). \end{aligned}$$

For $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$ define $z_A = P_{(n_0, \infty)}(x_{A_0 \cup A_1 \cup A})$ and $z_\emptyset = 0$. It then easily follows by (96) that Ψ is generated by the family $\{z_A : A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}\}$. Moreover, as $\max \text{supp}(z_A) = \max \text{supp}(x_{A_0 \cup A_1 \cup A})$, it is straightforward to check that Definition 9.5 (i) and (ii) are satisfied for Ψ . Whereas, observing that for all $A \in \mathcal{S}_{\beta\gamma} \cap [N]^{<\omega}$, with $A \neq \emptyset$ (which is the case when $l_1(A) > 0$), we have $\max \text{supp}(z_A) = \max \text{supp}(x_{A_0 \cup A_1 \cup A}) \geq n_0$, an

argument similar to the one used to obtain (95) also yields that Ψ satisfies Definition 9.5 (iii) and (iv). \square

The main result of this section is the following Refinement Argument.

Lemma 9.8. *Let $\alpha < \omega_1$, $M \in [\mathbb{N}]^\omega$, and A_0 be a subset of \mathbb{N} that is either empty or a singleton. Let also $\Phi : \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_\alpha \cap [M]^{<\omega}$ into X starting at A_0 . Then, for every $c < c(\Phi)$, there exists $N \in [M]^\omega$ and a semi-embedding $\tilde{\Phi} : \mathcal{S}_\alpha(A_0) \cap [N]^{<\omega} \rightarrow X$ of $\mathcal{S}_\alpha \cap [N]^{<\omega}$ into X starting at A_0 , that is c -refined.*

Proof. Put $\tilde{c} = (c(\Phi) + c)/2$. Let $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}\}$ be the vectors generating Φ and choose $\eta > 0$ with $\eta < c(\Phi) - \tilde{c}$. After shifting we can assume without loss of generality that $x_\emptyset = 0$. Set $\tilde{x}_\emptyset = 0$ and choose for each $A \in \mathcal{S}_\alpha(A_0) \cap [M]^{<\omega}$, with $A_0 \cup A \neq \emptyset$, a vector $\tilde{x}_{A_0 \cup A} \in B_X \cap c_{00}(E_j : j \in \mathbb{N})$ so that:

$$(a) \quad \|\tilde{x}_{A_0 \cup A} - x_{A_0 \cup A}\| < \eta/2, \text{ and } \max(A_0 \cup A) \leq \max \text{supp}(\tilde{x}_{A_0 \cup A}).$$

Moreover, recursively choose $\tilde{m}_1 < \dots < \tilde{m}_k < \dots$ in M so that for all k we have $\tilde{m}_{k+1} > \max \{ \max \text{supp}(\tilde{x}_{A_0 \cup A}) : A \in \mathcal{S}_\alpha(A_0) \cap [\{\tilde{m}_1, \dots, \tilde{m}_k\}] \}$. Define $\tilde{M} = \{\tilde{m}_k : k \in \mathbb{N}\}$ and $\tilde{\Phi} : \mathcal{S}_\alpha(A_0) \cap [\tilde{M}]^{<\omega} \rightarrow X$ so that for all $A \in \mathcal{S}_\alpha(A_0) \cap [\tilde{M}]^{<\omega}$ we have

$$\tilde{\Phi}(A) = \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) \tilde{x}_C.$$

By Lemma 9.4, $\tilde{\Phi}$ is a semi-embedding from $\mathcal{S}_\alpha \cap [\tilde{M}]^{<\omega}$ into X starting at A_0 for which $c(\tilde{\Phi}) > \tilde{c} > c$, and for which the conditions (i) and (ii) of Definition 9.5 are satisfied.

The goal is to find $N \in [\tilde{M}]^\omega$ so that, restricting $\tilde{\Phi}$ to $\mathcal{S}_\alpha(A_0) \cap [N]^{<\omega}$, Definition 9.5 (iii) and (iv) are satisfied as well. Put $M_0 = \tilde{M}$. Recursively, we will choose for every $k \in \mathbb{N}$ an infinite set $M_k \subset M_{k-1}$ so that for each $k \in \mathbb{N}$ the following conditions are met:

- (b) $\min(M_{k-1}) < \min(M_k)$, and putting $m_j = \min(M_j)$ for $j = 1, \dots, k-1$, then
- (c) for every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$, $r \in (0, 1]$, and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\{m_1, \dots, m_{k-1}\} \cup M_k]^{<\omega}$ with $B_1 < B_2$, we have

$$\|P_{(\max \text{supp}(\tilde{x}_{A_0 \cup A}), \infty)}(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2))\| \geq \tilde{c} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2),$$

- (d) for every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$, $r \in (0, 1]$, $B \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\{m_1, \dots, m_{k-1}\} \cup M_k]^{<\omega}$ we have

$$\|P_{(\max \text{supp}(\tilde{x}_{A_0 \cup A}), \infty)}(\tilde{\Phi}(A \cup B))\| \geq \frac{c}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C)$$

$$(\text{if } k = 1, [\{m_1, \dots, m_{k-1}\}] = \{\emptyset\}).$$

If we assume that such a sequence $(M_k)_k$ has been chosen, it is straightforward to check that $N = \{m_k : k \in \mathbb{N}\}$ is the desired set. In the case $k = 1$, for $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A = \emptyset$. Hence, if $A_0 = \emptyset$ then for all $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A_0 \cup A = \emptyset$, i.e. $l_1(A_0 \cup A) = 0$ and hence, (c) and (d) are always satisfied. Choosing M_1 satisfying (b) completes the first inductive step. If, on the other hand, A_0 is a singleton then for all $A \in \mathcal{S}_\alpha(A_0) \cap \{\emptyset\}$ we have $A_0 \cup A = A_0$, i.e. $l_1(A_0 \cup A) > 0$. This means that condition (c) and (d) are reduced to the case in which $A = \emptyset$. The choice of M_1 is done in the same manner as in the general inductive step and we omit it.

Assume that we have chosen, for some $k \geq 1$, infinite sets $M_k \subset M_{k-1} \subset \dots \subset M_1 \subset M_0$, so that (b), (c), and (d) are satisfied for all $1 \leq k' \leq k$. Observe that the inductive

assumption implies that it is enough to choose $M_{k+1} \in [M_k]^\omega$ satisfying (b), and the conditions (c) and (d) for sets $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_{k-1}\}]$ with $l_1(A_0 \cup A) > 0$ and $\max(A) = m_k = \min(M_k)$ (or $A = \emptyset$, in the case $k = 1$, and A_0 is a singleton). We set

$$\delta = \min \{ \zeta(\alpha, A_0 \cup A) : A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}] \text{ and } A_0 \cup A \neq \emptyset \},$$

$$\varepsilon = \frac{\delta}{30}(\tilde{c} - c),$$

$$d = \max \{ \max \text{supp}(\tilde{x}_{A_0 \cup A}) : A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_k\}] \},$$

and choose a finite ε -net R of the interval $(0, 1]$, with $1 \in R$, which also has the property that for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$ and all $j = 0, 1, 2, \dots, l_1(A_0 \cup A)$,

$$(97) \quad j \cdot \zeta(\alpha, A_0 \cup A) + \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) \in R.$$

Fix a finite $\frac{\varepsilon}{2}$ -net K of the unit ball of the finite dimensional space $\text{span}(E_j : 1 \leq j \leq d)$. For every $r \in R$ and $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$ we apply Proposition 2.14 to $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M_k]^{<\omega}$, and find an infinite subset \tilde{M}_{k+1} of M_k so that for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$ and $r \in R$, there exists $y_A^{(r)}$ in K so that

$$\|y_A^{(r)} - P_{[1,d]}(\tilde{\Phi}(A \cup B))\| < \frac{\varepsilon}{2}, \text{ for all } B \text{ in } \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\tilde{M}_{k+1}]^{<\omega}.$$

In particular, note that for all $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$ and $r \in R$, for any B_1, B_2 in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\tilde{M}_{k+1}]^{<\omega}$ we have

$$(98) \quad \|P_{[1,d]}(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2))\| < \varepsilon.$$

Using the property (iv) of $l_1(\cdot)$ in Lemma 3.7, we can pass to an infinite subset \widehat{M}_{k+1} of \tilde{M}_{k+1} , so that (b) is satisfied and moreover:

$$(99) \quad \zeta(\alpha, A_0 \cup A \cup B) < \varepsilon, \text{ if } A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_k\}], \text{ and}$$

$$B \in \mathcal{S}_\alpha(A_0 \cup A) \cap [M_{k+1}]^{<\omega} \text{ with } \#B > l_1(A_0 \cup A) > 0.$$

We will show that (c) is satisfied. To that end, fix $0 < r \leq 1$, $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_k\}]$ with $\max(A_0 \cup A) = m_k$ and $l_1(A_0 \cup A) > 0$, and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [\widehat{M}_{k+1}]^{<\omega}$ with $B_1 < B_2$. If both sets B_1 and B_2 are empty, then (c) trivially holds, as the right-hand side of the inequality has to be zero. Otherwise, $B_s \neq \emptyset$, where $s = 1$ or $s = 2$. Note that $\max(A) = m_k$, i.e. $A_0 \cup A \neq \emptyset$ and hence, since $l_1(A_0 \cup A) > 0$, putting $\tilde{C} = A_0 \cup A \cup \{\min(B_s)\}$, by the definition of δ we obtain $\zeta(\alpha, \tilde{C}) = \zeta(\alpha, A_0 \cup A) \geq \delta$. This easily yields:

$$\begin{aligned} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2) &= \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B_1} \zeta(\alpha, C) + \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B_2} \zeta(\alpha, C) \\ &\geq \delta = \frac{30\varepsilon}{\tilde{c} - c} \end{aligned}$$

Hence:

$$(100) \quad \varepsilon \leq \frac{\tilde{c} - c}{30} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2)$$

Arguing similarly, we obtain:

$$(101) \quad r \geq \sum_{C \preceq A_0 \cup A \cup B_s} \zeta(\alpha, C) \geq \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) + \zeta(\alpha, \tilde{C}) \geq \delta > \min(R).$$

Choose r_0 to be the maximal element of R with $r_0 \leq r$. Since $r_0 \leq r$, we can find \tilde{B}_1 and \tilde{B}_2 in $\mathcal{M}_\alpha^{(r_0)}(A_0 \cap A) \cap [\widehat{M}_{k+1}]^{<\omega}$ so that $\tilde{B}_1 \preceq B_1$ and $\tilde{B}_2 \preceq B_2$. We will show that

$$(102) \quad d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) < 3\varepsilon \text{ and } d_{1,\alpha}(A_0 \cup A \cup B_2, A_0 \cup A \cup \tilde{B}_2) < 3\varepsilon.$$

We shall only show that this is the case for B_1 , for B_2 the proof is identical. If $\tilde{B}_1 = B_1$, then there is nothing to prove and we may therefore assume that $\tilde{B}_1 \prec B_1$. Define $C_1 = \tilde{B}_1 \cup \{\min(B_1 \setminus \tilde{B}_1)\}$, $r_1 = \sum_{C \preceq A_0 \cup A \cup B_1} \zeta(\alpha, C)$, $\tilde{r}_1 = \sum_{C \preceq A_0 \cup A \cup \tilde{B}_1} \zeta(\alpha, C)$, and $r' = \sum_{C \preceq A_0 \cup A \cup C_1} \zeta(\alpha, C)$. The maximality of \tilde{B}_1 in $\mathcal{S}_\alpha^{(r_0)}(A_0 \cup A)$ implies

$$(103) \quad \tilde{r}_1 \leq r_0 < r' \leq r_1.$$

We first assume that $\#C_1 \leq l_1(A_0 \cup A)$. In this case however, by (97), we obtain that $r' = \sum_{C \preceq A_0 \cup A} \zeta(\alpha, C) + (\#C_1)\zeta(\alpha, A_0 \cup A)$ is in R . This contradicts the maximality of r_0 . We conclude that $\#C_1 > l_1(A_0 \cup A)$, which by (99) yields $r' - \tilde{r}_1 = \zeta(\alpha, A_0 \cup A \cup C_1) < \varepsilon$. Hence,

$$(104) \quad d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) = r_1 - \tilde{r}_1 = (r_1 - r') + (r' - \tilde{r}_1) < (r_1 - r_0) + \varepsilon < 3\varepsilon$$

We verify (c) now as follows

$$\begin{aligned} & \|P_{(d,\infty)}(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup B_2))\| \\ & \geq \|P_{(d,\infty)}(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2))\| \\ & \quad - \left(\|P_{(d,\infty)}(\tilde{\Phi}(A \cup B_1) - \tilde{\Phi}(A \cup \tilde{B}_1))\| + \|P_{(d,\infty)}(\tilde{\Phi}(A \cup B_2) - \tilde{\Phi}(A \cup \tilde{B}_2))\| \right) \\ & \geq \|P_{(d,\infty)}(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2))\| \\ & \quad - \left(d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup \tilde{B}_1) + d_{1,\alpha}(A_0 \cup A \cup B_2, A_0 \cup A \cup \tilde{B}_2) \right) \\ & > \|P_{(d,\infty)}(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2))\| - 6\varepsilon \\ & \geq \|\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\| - \|P_{[1,d]}(\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2))\| - 6\varepsilon \\ & \geq \|\tilde{\Phi}(A \cup \tilde{B}_1) - \tilde{\Phi}(A \cup \tilde{B}_2)\| - 7\varepsilon \\ & \geq c(\tilde{\Phi})d_{1,\alpha}(A_0 \cup A \cup \tilde{B}_1, A_0 \cup A \cup \tilde{B}_2) - 7\varepsilon > \tilde{c}d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2). \end{aligned}$$

Here the second inequality follows from (90), the third from (102), the fifth from (98), the sixth one from (91), and the last one from (100).

We will need to pass to a further subset of \widehat{M}_{k+1} to obtain (d). An application of the triangle inequality and (c) (for $k+1$) yield that for every $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$ with $l_1(A_0 \cup A)$, $r \in (0, 1]$ and $B_1, B_2 \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A)$, $B_1 < B_2$, it follows that

$$\|P_{(\max \text{supp}(\tilde{x}_{A_0 \cup A}), \infty)}(\tilde{\Phi}(A \cup B_1))\| \geq \frac{\tilde{c}}{2} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2),$$

or

$$\|P_{(\max \text{supp}(\tilde{x}_{A_0 \cup A}), \infty)}(\tilde{\Phi}(A \cup B_2))\| \geq \frac{\tilde{c}}{2} d_{1,\alpha}(A_0 \cup A \cup B_1, A_0 \cup A \cup B_2).$$

We may pass therefore to a further infinite subset M_{k+1} of \widehat{M}_{k+1} , so that for any $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, m_2, \dots, m_k\}]$, with $l_1(A_0 \cup A) > 0$ and $\max(A_0 \cup A) = m_k$, any $r \in R$, and for any B in $\mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M_{k+1}]^{<\omega}$ we have

$$(105) \quad \|P_{(d,\infty)}\tilde{\Phi}(A \cup B)\| \geq \frac{\tilde{c}}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C).$$

We will show that (d) is satisfied (meaning that (105) does not only hold if $r \in R$). Fix an arbitrary $r \in (0, 1]$, $A \in \mathcal{S}_\alpha(A_0) \cap [\{m_1, \dots, m_k\}]$, with $\max(A_0 \cup A) = m_k$ and $l_1(A_0 \cup A) > 0$, and $B \in \mathcal{M}_\alpha^{(r)}(A_0 \cup A) \cap [M_{k+1}]^{<\omega}$. If B is empty, then the right-hand side of the inequality in (d) is zero and hence the conclusion holds. Arguing identically as in (100) we obtain

$$\varepsilon \leq \frac{\tilde{c} - c}{30} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C)$$

and arguing as in (101) we obtain $r > \min(R)$, so we may choose the maximal element r_0 of R with $r_0 \leq r$, and \tilde{B} the maximal element of $\mathcal{M}_\alpha^{(r_0)}(A_0 \cup A) \cap [M_{k+1}]^{<\omega}$ with $\tilde{B} \preceq B$. The argument yielding (102), also yields

$$d_{1,\alpha}(A_0 \cup A \cup B, A_0 \cup A \cup \tilde{B}) = \sum_{A_0 \cup A \cup \tilde{B} \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C) < 3\varepsilon.$$

Arguing in a very similar manner as above:

$$\begin{aligned} \|P_{(d,\infty)}\tilde{\Phi}(A \cup B)\| &\geq \|P_{(d,\infty)}\tilde{\Phi}(A \cup \tilde{B})\| - \|P_{(d,\infty)}\tilde{\Phi}(A \cup B) - P_{(d,\infty)}\tilde{\Phi}(A \cup \tilde{B})\| \\ &\geq \frac{\tilde{c}}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup \tilde{B}} \zeta(\alpha, C) - 3\varepsilon \\ &\geq \frac{\tilde{c}}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C) - \left(3 + \frac{3\tilde{c}}{2}\right) \varepsilon \\ &\geq \frac{\tilde{c}}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C) - \left(\frac{\tilde{c} - c}{2}\right) \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C) \\ &= \frac{c}{2} \sum_{A_0 \cup A \prec C \preceq A_0 \cup A \cup B} \zeta(\alpha, C). \end{aligned}$$

□

10. SOME FURTHER OBSERVATION ON THE SCHREIER FAMILIES

In this section β will be a fixed ordinal of the form $\beta = \omega^{\omega^\xi}$, with $1 \leq \xi < \omega_1$.

10.1. Analysis of a maximal set B in $\mathcal{S}_{\beta\gamma}$. Recall that by Proposition 2.6, for every $\gamma \leq \beta$ there exists a sequence $\eta(\gamma, n)$ of ordinal numbers increasing to γ , so that $\lambda(\beta\gamma, n) = \beta\eta(\gamma, n)$ (recall that $\eta(\gamma, n)$ may also depend on β).

For every $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$ we define a family of subsets of B , which we shall call the β -analysis of B , and denote by $\mathcal{A}_{\beta,\gamma}(B)$. The definition is done recursively on γ .

For $\gamma = 1$, set

$$(106a) \quad \mathcal{A}_{\beta,\gamma}(B) = \{B\}.$$

Let $\gamma < \beta$ and assume that $\mathcal{A}_{\beta,\gamma}(B)$ has been defined for all $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. For $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \text{MAX}(\mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}])$ there are (uniquely defined) $B_1 < \dots < B_\ell$ in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ with $\{\min B_j : j = 1, \dots, \ell\}$ in $\text{MAX}(\mathcal{S}_\beta)$ so that $B = \bigcup_{j=1}^\ell B_j$. Set

$$(106b) \quad \mathcal{A}_{\beta,\gamma+1}(B) = \{B\} \cup \left(\bigcup_{j=1}^\ell \mathcal{A}_{\beta,\gamma}(B_j) \right).$$

Let $\gamma \leq \beta$ be a limit ordinal and assume that $\mathcal{A}_{\beta,\gamma'}(B)$ has been defined for all $\gamma' < \gamma$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma'})$. If now $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma)})$, then $B \in \text{MAX}(\mathcal{S}_{\beta\eta(\gamma, \min(B))})$. Set

$$(106c) \quad \mathcal{A}_{\beta,\gamma}(B) = \mathcal{A}_{\beta,\eta(\gamma, \min(B))}(B).$$

Remark 10.1. Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. The following properties are straightforward consequences of the definition of $\mathcal{A}_{\beta,\gamma}(B)$ and a transfinite induction.

- (i) The set $\mathcal{A}_{\beta,\gamma}(B)$ is a tree when endowed with “ \supset ”.
- (ii) For C, D in $\mathcal{A}_{\beta,\gamma}(B)$ that are incomparable with respect to inclusion, we have either $C < D$, or $D < C$.
- (iii) The minimal elements (with respect to inclusion) of $\mathcal{A}_{\beta,\gamma}(B)$ are in \mathcal{S}_β .
- (iv) If $D \in \mathcal{A}_{\beta,\gamma}(B)$ is a non-minimal element, then its direct successors $(D_j)_{j=1}^\ell$ in $\mathcal{A}_{\beta,\gamma}(B)$ can be enumerated so that $D_1 < \dots < D_\ell$ and $D = \cup_{j=1}^\ell D_j$.

10.2. Components of a set A in $\mathcal{S}_{\beta\gamma}$. We recursively define for all non-empty sets $A \in \mathcal{S}_{\beta\gamma}$ and $\gamma \leq \beta$, a natural number $s(\beta, \gamma, A)$ and subsets $\text{Cp}_{\beta,\gamma}(A, 1), \dots, \text{Cp}_{\beta,\gamma}(A, s(\beta, \gamma, A))$ of A . We will call $(\text{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ the *components of A in $\mathcal{S}_{\beta\gamma}$, with respect to \mathcal{S}_β* .

If $\gamma = 1$, i.e. A is a non-empty set in \mathcal{S}_β , define

$$(107a) \quad s(\beta, \gamma, A) = 1 \text{ and } \text{Cp}_{\beta,\gamma}(A, 1) = A.$$

Let $\gamma \leq \beta$ and assume that $(\text{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ has been defined for all non-empty sets A in $\mathcal{S}_{\beta\gamma}$. If now A is a non-empty set in $\mathcal{S}_{\beta(\gamma+1)} = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$, then there are non-empty sets $A_1 < A_2 < \dots < A_d$ in $\mathcal{S}_{\beta\gamma}$ so that:

- (i) $A = \cup_{i=1}^d A_i$,
- (ii) $\{\min A_i : i = 1, \dots, d\}$ is in \mathcal{S}_β , and
- (iii) the sets A_1, \dots, A_{d-1} are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$.

Note that the set A_d may or may not be in $\text{MAX}(\mathcal{S}_{\beta\gamma})$. It may also be the case that $d = 1$, which in particular happens when $A \in \mathcal{S}_{\beta\gamma}$. Set $\bar{A} = A_d$, which is always non-empty, and we define

$$(107b) \quad \begin{aligned} s(\beta, \gamma + 1, A) &= s(\beta, \gamma, \bar{A}) + 1, \\ \text{Cp}_{\beta,\gamma+1}(A, 1) &= \cup_{i < d} A_i, \text{ and} \\ \text{Cp}_{\beta,\gamma+1}(A, i) &= \text{Cp}_{\beta,\gamma}(\bar{A}, i-1), \text{ if } 2 \leq i \leq s(\beta, \gamma + 1, A). \end{aligned}$$

Note that in the case $d = 1$, $\text{Cp}_{\beta,\gamma+1}(A, 1)$ is the empty set.

Let $\gamma \leq \beta$ be a limit ordinal and assume that $(\text{Cp}_{\beta,\gamma'}(A, i))_{i=1}^{s(\beta,\gamma',A)}$ has been defined for all $\gamma' < \gamma$ and non-empty sets A in $\mathcal{S}_{\beta\gamma'}$. If A is a non-empty set in $\mathcal{S}_{\beta\gamma}$, then $A \in \mathcal{S}_{\beta\eta(\gamma, \min(A))}$ and we define

$$(107c) \quad \begin{aligned} s(\beta, \gamma, A) &= s(\beta, \eta(\gamma, \min(A)), A) \text{ and} \\ \text{Cp}_{\beta,\gamma}(A, i) &= \text{Cp}_{\beta,\eta(\gamma, \min(A))}(A, i) \text{ for } i = 1, 2, \dots, s(\beta, \gamma, A). \end{aligned}$$

Remark. Let $\gamma \leq \beta$ and $A \in \mathcal{S}_{\beta\gamma} \setminus \{\emptyset\}$. The following properties follow easily from the definition of $(\text{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ and a transfinite induction on γ .

- (i) $A = \cup_{i=1}^{s(\beta,\gamma,A)} \text{Cp}_{\beta,\gamma}(A, i)$.
- (ii) For $1 \leq i < j \leq s(\beta, \gamma, A)$ so that both $\text{Cp}_{\beta,\gamma}(A, i)$ and $\text{Cp}_{\beta,\gamma}(A, j)$ are non-empty, we have $\text{Cp}_{\beta,\gamma}(A, i) < \text{Cp}_{\beta,\gamma}(A, j)$.
- (iii) $\text{Cp}(A, s(\beta, \gamma(A))) \neq \emptyset$.

Lemma 10.2. *Let ξ and γ be countable ordinal numbers with $\gamma \leq \beta = \omega^{\omega^\xi}$. Let also B be a set in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ and $\emptyset \prec A \preceq B$. If $\mathcal{A}_{\beta,\gamma}(B)$ is the β -analysis of B and $(\text{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)}$ are the components of A in $\mathcal{S}_{\beta\gamma}$ with respect to \mathcal{S}_β , then there exists a maximal chain $B = D_1(A) \supsetneq D_2(A) \supsetneq \cdots \supsetneq D_{s(\beta,\gamma,A)}(A)$ in $\mathcal{A}_{\beta,\gamma}(B)$ satisfying the following:*

- (i) $\text{Cp}_{\beta,\gamma}(A, i) \preceq D_i(A)$ for $1 \leq i \leq s(\beta, \gamma, A)$, and
- (ii) if $1 \leq i < s(\beta, \gamma, A)$ then $\text{Cp}_{\beta,\gamma}(A, i) \subsetneq D_i(A)$.

Proof. We prove the statement by transfinite induction on $1 \leq \gamma \leq \beta$. If $\gamma = 1$, then $\mathcal{A}_{\beta,\gamma}(B) = \{B\}$ and $(\text{Cp}_{\beta,\gamma}(A, i))_{i=1}^{s(\beta,\gamma,A)} = \{A\}$, and our claim follows trivially.

Let $\gamma < \beta$ and assume that the statement holds for all non-empty $A \in \mathcal{S}_{\beta\gamma}$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$ with $A \preceq B$. Let now A be a non-empty set in $\mathcal{S}_{\beta(\gamma+1)}$ and $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)})$. If $B = \cup_{j=1}^\ell B_j$, where $B_1 < \cdots < B_\ell$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ with $\{\min(B_j) : 1 \leq j \leq \ell\} \in \text{MAX}(\mathcal{S}_\beta)$, then by (106b) we obtain $\mathcal{A}_{\beta,\gamma+1}(B) = \{B\} \cup (\cup_{j=1}^\ell \mathcal{A}_{\beta,\gamma}(B_j))$. Define

$$d = \max\{1 \leq j \leq \ell : B_j \cap A \neq \emptyset\}, \quad A_i = B_i \text{ for } 1 \leq i < d, \text{ and } A_d = A \cap B_d.$$

Then, by (107b), letting $\bar{A} = A_d$, we obtain $s(\beta, \gamma + 1, A) = s(\beta, \gamma, \bar{A}) + 1$, $\text{Cp}_{\beta,\gamma+1}(A, 1) = \cup_{i < d} A_i$, and $\text{Cp}_{\beta,\gamma+1}(A, i) = \text{Cp}_{\beta,\gamma}(\bar{A}, i - 1)$, for $2 \leq i \leq s(\beta, \gamma + 1, A)$. Apply the inductive assumption to $A_d = \bar{A} \preceq B_d$, to find a maximal chain $B_d = D_1(\bar{A}) \supsetneq \cdots \supsetneq D_{s(\beta,\gamma,\bar{A})}(\bar{A})$ in $\mathcal{A}_{\beta\gamma}(B_d)$ satisfying (i) and (ii) with respect to $(\text{Cp}_{\beta,\gamma}(\bar{A}, i))_{i=1}^{s(\beta,\gamma,\bar{A})}$. Define

$$(108) \quad D_1(A) = B, \text{ and } D_i(A) = D_{i-1}(\bar{A}), \text{ for } 2 \leq i \leq s(\beta, \gamma, A).$$

Clearly, $(D_i(A))_{i=1}^{s(\beta,\gamma,\bar{A})}$ is a maximal chain in $\mathcal{A}_{\beta,\gamma+1}(B)$. It remains to verify that (i) and (ii) are satisfied, with respect to $(\text{Cp}_{\beta,\gamma+1}(A, i))_{i=1}^{s(\beta,\gamma+1,A)}$. Assertions (i) and (ii), in the case $i = 1$, follow trivially from $\text{Cp}_{\beta,\gamma+1}(A, 1) = \cup_{j < d} B_j = \cup_{j < d} B_j \prec \cup_{j < d} B_j \preceq B$. Assertion (i) and (ii) in the case $i \neq 1$, follow easily from the inductive assumption and $\text{Cp}_{\beta,\gamma+1}(A, i) = \text{Cp}_{\beta,\gamma}(\bar{A}, i - 1)$, for $2 \leq i \leq s(\beta, \gamma + 1, A)$.

To conclude the proof, the case in which $\gamma \leq \beta$ is a limit ordinal number so that the conclusion is satisfied for all $\gamma' < \gamma$, we just observe that the result is an immediate consequence of (106c) and (107c). \square

For the next result recall the definition of the doubly indexed fine Schreier families $\mathcal{F}_{\beta,\gamma}$ introduced in Subsection 2.3.

Lemma 10.3. *Let $\gamma \leq \beta$. Then for all $A \in \mathcal{S}_{\beta\gamma}$, with $\text{Cp}_{\beta\gamma}(A, i) \neq \emptyset$ for $1 \leq i \leq s(\beta, \gamma, A)$, we have*

$$(109) \quad \{\min(\text{Cp}_{\beta,\gamma}(A, i)) : 1 \leq i \leq s(\beta, \gamma, A)\} \in \text{MAX}(\mathcal{F}_{\beta,\gamma}).$$

Proof. We prove this statement by induction on γ . If $\gamma = 1$ and $A \in \mathcal{S}_\beta$ satisfying the assumptions of this Lemma, then $A = \text{Cp}_{\beta,1}(A, 1) \neq \emptyset$ and hence the result easily follows from $\text{MAX}(\mathcal{F}_{\beta,1}) = \{\{n\} : n \in \mathbb{N}\}$.

Assume that the result holds for some $\gamma < \beta$ and let $A \in \mathcal{S}_{\beta(\gamma+1)}$, with $\text{Cp}_{\beta\gamma}(A, i) \neq \emptyset$ for $1 \leq i \leq s(\beta, \gamma + 1, A)$. By the inductive assumption and (107b) we obtain that $B = \{\min(\text{Cp}_{\beta,\gamma}(A, i)) : 2 \leq i \leq s(\beta, \gamma + 1, A)\} \in \text{MAX}(\mathcal{F}_{\beta,\gamma})$. We claim that $\tilde{A} = \{\min(\text{Cp}_{\beta,\gamma+1}(A, 1))\} \cup B \in \text{MAX}(\mathcal{F}_{\beta,\gamma+1})$. Indeed, if this is not the case, then by the spreading property of $\mathcal{F}_{\beta,\gamma+1}$ there is $C \in \mathcal{F}_{\beta,\gamma+1}$ with $\tilde{A} \prec C$. Then, $B \prec \tilde{C} = C \setminus \{\min(\text{Cp}_{\beta,\gamma+1}(A, 1))\}$. It follows by $\mathcal{F}_{\beta,\gamma+1} = \mathcal{F}_{\beta,1} \sqcup \mathcal{F}_{\beta,\gamma}$, that $\tilde{C} \in \mathcal{F}_{\beta,\gamma}$. The maximality of B yields a contradiction.

Assume now that $\gamma \leq \beta$ is a limit ordinal number so that the conclusion holds for all $\tilde{\gamma} < \gamma$. Let $A \in \mathcal{S}_{\beta\gamma}$ so that $\text{Cp}_{\beta,\gamma}(A, i) \neq \emptyset$ for $1 \leq i \leq s(\beta, \gamma, A)$. Note that $A \in \mathcal{S}_{\beta\eta(\gamma, \min(A))}$ and by (107c) and the inductive assumption we have $\tilde{A} = \{\min(\text{Cp}_{\beta,\gamma}(A, i)) : 1 \leq i \leq s(\beta, \gamma, A)\} \in \text{MAX}(\mathcal{F}_{\beta, \eta(\gamma, \min(A))})$. By (27) we obtain $\tilde{A} \in \text{MAX}(\mathcal{F}_{\beta, \gamma})$. \square

For $\gamma \leq \beta$ and a set B in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ we define

$$(110) \quad \mathcal{E}_{\beta,\gamma}(B) = \{\emptyset \prec A \preceq B : \text{Cp}_{\beta,\gamma}(A, i) \neq \emptyset \text{ for } 1 \leq i \leq s(\beta, \gamma, A)\}.$$

Lemma 10.4. *Let $\gamma < \beta$. If B is in $\text{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \mathcal{S}_{\beta}[\mathcal{S}_{\beta\gamma}]$ and $B = \cup_{j=1}^{\ell} B_j$, where $B_1 < \dots < B_{\ell}$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ and $\{\min(B_j) : 1 \leq j \leq \ell\} \in \text{MAX}(\mathcal{S}_{\beta})$, then*

$$\{A : A \preceq B \text{ and } A \notin \mathcal{E}_{\beta(\gamma+1)}(B)\} = \{A : A \preceq B_1\} \cup \left(\bigcup_{m=2}^{\ell} \left\{ \left(\bigcup_{j=1}^{m-1} B_j \right) \cup A : A \preceq B_m \text{ and } A \notin \mathcal{E}_{\beta,\gamma}(B_m) \right\} \right).$$

Proof. Let $\emptyset \neq D \preceq B$. Define $m = \max\{1 \leq j \leq \ell : D \cap B_j \neq \emptyset\}$ and $A = B_m \cap D$. Note that $D = (\cup_{j < m} B_j) \cup A$, where $\cup_{j < m} B_j = \emptyset$ if $m = 1$. By (107b) we obtain, $s(\beta, \gamma+1, D) = s(\beta, \gamma, A) + 1$, $\text{Cp}_{\beta,\gamma+1}(D, 1) = \cup_{j < m} B_j$ and $\text{Cp}_{\beta,\gamma+1}(D, i) = \text{Cp}_{\beta,\gamma}(A, i-1)$ for $2 \leq i \leq s(\beta, \gamma+1, D)$.

Observe that $\text{Cp}_{\beta,\gamma+1}(D, 1) = \emptyset$ if and only if $m = 1$, i.e. $A = D \preceq B_1$. On the other hand, if $\text{Cp}_{\beta,\gamma+1}(D, 1) \neq \emptyset$, then for some $2 \leq i \leq s(\beta, \gamma+1, D)$, we have $\text{Cp}_{\beta,\gamma+1}(D, i) = \emptyset$, if and only if $\text{Cp}_{\beta,\gamma}(A, i-1) = \emptyset$. These observations yield our claim. \square

Remark. Under the assumptions of Lemma 10.4, if for $1 \leq j \leq \ell - 1$ we define

$$\mathcal{E}_{\beta,\gamma+1}^{(j)}(B) = \{A \in \mathcal{E}_{\beta,\gamma+1}(B) : \text{Cp}_{\beta,\gamma}(A, 1) = \cup_{i \leq j} B_i\},$$

then, using a similar argument as the one used in the proof of Lemma 10.4, we obtain

$$\mathcal{E}_{\beta,\gamma+1}^{(j)}(B) = \left\{ (\cup_{i=1}^j B_i) \cup C : C \in \mathcal{E}_{\beta,\gamma}(B_{j+1}) \right\} \text{ and } \mathcal{E}_{\beta(\gamma+1)}(B) = \cup_{j=1}^{\ell-1} \mathcal{E}_{\beta,\gamma+1}^{(j)}.$$

Remark. Using the fact $\mathcal{S}_1 \subset \mathcal{S}_{\alpha}$ that for all countable ordinal numbers α , and that $\text{MAX}(\mathcal{S}_1) = \{F \subset \mathbb{N} : \min(F) = \#F\}$, it is easy to verify that for all $F \in \text{MAX}(\mathcal{S}_{\alpha})$ we have $\max(F) \geq 2\min(F) - 1$. In particular, if $B_1 < B_2$ are both in $\text{MAX}(\mathcal{S}_{\alpha})$, then

$$(111) \quad 2\min(B_1) \leq \min(B_2).$$

Lemma 10.5. *Let $\gamma \leq \beta$ and let B be a set in $\text{MAX}(\mathcal{S}_{\beta\gamma})$. Then*

$$(112) \quad \sum_{\substack{A \prec B \\ A \notin \mathcal{E}_{\beta,\gamma}(B)}} \zeta(\beta\gamma, A) < \frac{2}{\min(B)}.$$

Proof. We prove the statement by transfinite induction for all $1 \leq \gamma \leq \beta$. If $\gamma = 1$, then the complement of $\mathcal{E}_{\beta,1}(B)$ only contains the empty set and the result trivially holds.

Let $\gamma < \beta$ and assume that the statement holds for all $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. Let $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)})$. Let $B_1 < \dots < B_{\ell}$ in $\text{MAX}(\mathcal{S}_{\beta\gamma})$ so that $\{\min(B_j) : 1 \leq j \leq \ell\} \in \text{MAX}(\mathcal{S}_{\beta})$ and $B = \cup_{j=1}^{\ell} B_j$. For $m = 1, \dots, \ell$, define $D_m = \{\min(B_j) : 1 \leq j \leq m\}$. Proposition 3.4 implies the following: if $C \preceq B$, $m = \max\{1 \leq j \leq \ell : C \cap B_j \neq \emptyset\}$ and $A = C \cap B_m$, then

$\zeta(\beta(\gamma+1), C) = \zeta(\beta, D_m)\zeta(\beta\gamma, A)$. We combine this fact with Lemma 10.4, (111), and (33) to obtain the following:

$$\begin{aligned}
\sum_{\substack{A \preceq B \\ A \notin \mathcal{E}_{\beta, \gamma}(B)}} \zeta(\beta\gamma, A) &= \zeta(\beta, D_1) \sum_{A \preceq B_1} \zeta(\beta\gamma, A) + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \sum_{\substack{A \preceq B_j \\ A \notin \mathcal{E}_{\beta, \gamma}(B_j)}} \zeta(\beta\gamma, A) \\
&< \zeta(\beta, D_1) + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{2}{\min(B_j)} \\
&\leq \frac{1}{\min(B_1)} + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{2}{\min(B_j)} \\
&\leq \frac{1}{\min(B_1)} + \sum_{j=2}^{\ell} \zeta(\beta, D_j) \frac{1}{\min(B_1)} \leq \frac{2}{\min(B_1)} = \frac{2}{\min(B)}.
\end{aligned}$$

If $\gamma \leq \beta$ is a limit ordinal number so that the conclusion is satisfied for all $\gamma' < \gamma$, we just observe that the result is an immediate consequence of (107c) and (30). \square

Lemma 10.6. *Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. If $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta, \gamma}(B)$, $(D_k(A^{(1)}))_{k=1}^{s(\beta, \gamma, A^{(1)})}$ and $(D_k(A^{(2)}))_{k=1}^{s(\beta, \gamma, A^{(2)})}$ are the maximal chains of $\mathcal{A}_{\beta, \gamma}(B)$ provided by Lemma 10.2, and $1 \leq i \leq \min\{s(\beta, \gamma, A^{(1)}), s(\beta, \gamma, A^{(2)})\}$ is such that $D_i(A^{(1)}) = D_i(A^{(2)})$, then we have $D_j(A^{(1)}) = D_j(A^{(2)})$ for all $1 \leq j < i$.*

Proof. As $(D_k(A^{(1)}))_{k=1}^{s(\beta, \gamma, A^{(1)})}$ and $(D_k(A^{(2)}))_{k=1}^{s(\beta, \gamma, A^{(2)})}$ are both maximal chains of $\mathcal{A}_{\beta, \gamma}(B)$ with $D_i(A^{(1)}) = D_i(A^{(2)})$, the result follows from Remark 10.1 (i). \square

Lemma 10.7. *Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. If $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta, \gamma}(B)$, and $1 \leq i \leq \min\{s(\beta, \gamma, A^{(1)}), s(\beta, \gamma, A^{(2)})\}$, then $D_i(A^{(1)}) = D_i(A^{(2)})$ if and only if $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, i)) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, i))$, where $(D_k(A^{(1)}))_{k=1}^{s(\beta, \gamma, A^{(1)})}$ and $(D_k(A^{(2)}))_{k=1}^{s(\beta, \gamma, A^{(2)})}$ are the maximal chains of $\mathcal{A}_{\beta, \gamma}(B)$ provided by Lemma 10.2.*

Proof. Assume that $D_i(A^{(1)}) = D_i(A^{(2)})$. Lemma 10.2 (i) and the assumptions $\text{Cp}(A^{(1)}, i) \neq \emptyset$, and $\text{Cp}(A^{(2)}, i) \neq \emptyset$ yield $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, i)) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, i))$. For the converse let $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta, \gamma}(B)$ with $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, i)) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, i))$. Since all elements of $\mathcal{A}_{\beta, \gamma}(B)$ either compare with respect to “ \subset ”, or are disjoint, Lemma 10.2 (i) and $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, i)) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, i))$ imply that either $D_i(A^{(1)}) \subset D_i(A^{(2)})$, or $D_i(A^{(2)}) \subset D_i(A^{(1)})$. We assume the first, and towards a contradiction assume that $D_i(A^{(1)}) \subsetneq D_i(A^{(2)})$. The maximality of $(D_j(A^{(2)}))_{j=1}^{s(\beta, \gamma, A^{(2)})}$ in $\mathcal{A}_{\beta, \gamma}(B)$ implies that there is $1 \leq j < i$, so that $D_j(A^{(2)}) = D_i(A^{(1)})$. As $A_{\beta, \gamma}^{(2)}(j) \neq \emptyset$, we obtain by Lemma 10.2 (i) $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, i)) = \min(D_i(A^{(1)})) = \min(D_j(A^{(2)})) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, j)) < \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, i))$ which is a contradiction. \square

10.3. Special families of convex combinations.

Definition 10.8. Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. A family of non-negative numbers $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(B), 1 \leq k \leq s(\beta, \gamma, A)\}$, is called a (β, γ) -special family of convex combinations for B if the following are satisfied:

- (i) $\sum_{k=1}^{s(\beta, \gamma, A)} r(A, k) = 1$ for all $A \in \mathcal{E}_{\beta, \gamma}(B)$ and
- (ii) If $A^{(1)}, A^{(2)}$ are both in $\mathcal{E}_{\beta, \gamma}(B)$, $(D_k(A^{(1)}))_{k=1}^{s(\beta, \gamma, A^{(1)})}$ and $(D_k(A^{(2)}))_{k=1}^{s(\beta, \gamma, A^{(2)})}$ are the maximal chains in $\mathcal{A}_{\beta, \gamma}(B)$ provided by Lemma 10.2, and for some k we have $D_k(A^{(1)}) = D_k(A^{(2)})$, then $r(A^{(1)}, k) = r(A^{(2)}, k)$.

Remark. Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$ and let $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(B), k = 1, 2, \dots, s(\beta, \gamma, A)\}$ be a family of (β, γ) -special convex combinations for B . For $A \in \mathcal{E}_{\beta, \gamma}(B)$, let $(D_k(A))_{k=1}^{s(\beta, \gamma, A)}$ be the maximal chain in $\mathcal{A}_{\beta, \gamma}(B)$ provided by Lemma 10.2, and let $(A(i))_{i=1}^{s(\beta, \gamma, A)}$ be the components of A in $\mathcal{S}_{\beta\gamma}$.

By construction, $D_1(A) = B$ for all $A \in \mathcal{E}_{\beta, \gamma}(B)$, and thus $r(A, 1)$ does not depend on A . Secondly $D_2(A)$, only depends on $A(1)$, thus $r(A^{(1)}, 2) = r(A^{(2)}, 2)$ if $\text{Cp}_{\beta, \gamma}(A^{(1)}, 1) = \text{Cp}_{\beta, \gamma}(A^{(2)}, 1)$, for any $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta, \gamma}(B)$. We can continue and inductively we observe that for all $k \leq \min(s(\beta, \gamma, A^{(1)}), s(\beta, \gamma, A^{(2)}))$ if $\text{Cp}_{\beta, \gamma}(A^{(1)}, i) = \text{Cp}_{\beta, \gamma}(A^{(2)}, i)$, for all $i = 1, 2, \dots, k-1$, then $r(A^{(1)}, k) = r(A^{(2)}, k)$.

Let $\gamma \leq \beta$ and $B \in \text{MAX}(\mathcal{S}_{\beta\gamma})$. If γ is a limit ordinal number, then $B \in \text{MAX}(\mathcal{S}_{\beta\eta(\gamma, \min(B))})$ and any (β, γ) -special family of convex combinations $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(B), 1 \leq k \leq s(\beta, \gamma, A)\}$ is also a $(\beta, \eta(\gamma, \min(B)))$ -special family of convex combinations, as $\mathcal{E}_{\beta, \gamma}(B) = \mathcal{E}_{\beta, \eta(\gamma, \min(B))}(B)$ and for $A \in \mathcal{E}_{\beta, \eta(\gamma, \min(B))}(B)$ we have $s(\beta, \eta(\gamma, \min(B)), A) = s(\beta, \gamma, A)$.

Lemma 10.9. *We are given $\gamma < \beta$, $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)})$, and a $(\beta, \gamma+1)$ -special family of convex combinations $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma+1}(B), 1 \leq k \leq s(\beta, \gamma+1, A)\}$. Assume that for some $D \in \mathcal{E}_{\beta, \gamma+1}(B)$ (and hence for all of them) we have $r(D, 1) < 1$. Let $B = \bigcup_{j=1}^d B_j$, where $B_1 < \dots < B_d$ are the immediate predecessors of B in $\mathcal{A}_{\beta\gamma}(B)$. For every $1 \leq j < d$ consider the family $\{r^{(j)}(C, k) : C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})\}$, with*

$$r^{(j)}(C, k) = \frac{1}{1 - r(D, 1)} r\left(\left(\bigcup_{i=1}^j B_i\right) \cup C, k+1\right)$$

for $k = 1, \dots, s(\beta, \gamma, C) = s(\beta, \gamma, (\bigcup_{i=1}^j B_i) \cup C) - 1$. Then $\{r^{(j)}(C, k) : C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})\}$ is a (β, γ) -special family of convex combinations.

Proof. By (107b), if $C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})$ then $A = (\bigcup_{i=1}^j B_i) \cup C \in \mathcal{E}_{\beta, \gamma+1}(B)$ and $s(\beta, \gamma+1, A) = s(\beta, \gamma, C) + 1$, which implies that Definition 10.8 (i) is satisfied. To see that (ii) holds, let $C^{(1)}, C^{(2)}$ be in $\mathcal{E}_{\beta, \gamma}(B_{j+1})$ so that for some k we have $D_k(C^{(1)}) = D_k(C^{(2)})$. Then by Lemma 10.7 we have $\min(\text{Cp}_{\beta, \gamma}(C^{(1)}, k) = \min(\text{Cp}_{\beta, \gamma}(C^{(2)}, k)$. Setting $A^{(1)} = (\bigcup_{i=1}^j B_i) \cup C^{(1)}$ and $A^{(2)} = (\bigcup_{i=1}^j B_i) \cup C^{(2)}$, by (107b) we obtain $\text{Cp}_{\beta, \gamma+1}(A^{(1)}, k+1) = \text{Cp}_{\beta, \gamma}(C^{(1)}, k)$ and $\text{Cp}_{\beta, \gamma}(A^{(2)}, k+1) = \text{Cp}_{\beta, \gamma}(C^{(2)}, k)$, i.e. $\min(\text{Cp}_{\beta, \gamma}(A^{(1)}, k+1)) = \min(\text{Cp}_{\beta, \gamma}(A^{(2)}, k+1))$. By Lemma 10.7 we obtain $D_{k+1}(A^{(1)}) = D_{k+1}(A^{(2)})$ and therefore $r(A^{(1)}, k+1) = r(A^{(2)}, k+1)$, which yields that $r^{(j)}(C^{(1)}, k) = r^{(j)}(C^{(2)}, k)$. \square

11. CONCLUSION OF THE PROOF OF THEOREMS A AND C

Again, we fix $\xi < \omega_1$ and put $\beta = \omega^\xi$. We secondly assume that X is a Banach space X with a bimonotone FDD (F_j) . By the main result in [33] every reflexive Banach space X embeds into a reflexive Banach space Z with basis, so that $\text{Sz}(Z) = \text{Sz}(X)$ and $\text{Sz}(Z^*) = \text{Sz}(X^*)$. The coordinate projections on finitely or cofinitely many coordinates are denoted by P_A (see Section 9 after Definition 9.5).

Definition 11.1. Let $\gamma \leq \beta$, $M \in [N]^\omega$ and A_0 be a subset of \mathbb{N} that is either empty or a singleton. Let also $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X , starting after A_0 , that is c -refined, for some $0 < c \leq 1$. Let $\{x_\emptyset\} \cup \{x_{A_0 \cup A} : A \in \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega}\}$ be the family generating Φ (recall that notation from the Remark after Definition 9.2).

Let $E \in \text{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega})$. For $A \preceq E$ recall the definition of $s(\beta, \gamma, A)$ and of $(\text{Cp}_{\beta, \gamma}(A, i))_{i=1}^{s(\beta, \gamma, A)}$. Recall also from (110) $\mathcal{E}_{\beta, \gamma}(A_0 \cup E) = \{\emptyset \prec A \preceq A_0 \cup E : \text{Cp}_{\beta, \gamma}(A, i) \neq \emptyset, \text{ for } i = 1, 2, \dots, s(\beta, \gamma, A)\}$.

For each $A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)$ we will write x_A as a sum of a block sequence

$$(113) \quad x_A = \sum_{k=1}^{s(\beta, \gamma, A)} x_{\Phi, A}^{(k)},$$

with $x_{\Phi, A}^{(k)} = P_{I(A, k)}(x_A)$, for $k = 1, 2, \dots, s(\beta, \gamma, A)$, where $I(A, 1) = [1, \max \text{supp}(x_{\text{Cp}_{\beta, \gamma}(A, 1)})]$ and $I(A, k) = (\max \text{supp}(x_{\cup_{i=1}^{k-1} \text{Cp}_{\beta, \gamma}(A, i)}), \max \text{supp}(x_{\cup_{i=1}^k \text{Cp}_{\beta, \gamma}(A, i)}])$ for $1 < k < s(\beta, \gamma, A)$.

We call $((x_{\Phi, A}^{(k)})_{k=1}^{s(\beta, \gamma, A)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)}$ the *block step decomposition of E with respect to Φ* .

Remark. Let $M \in [\mathbb{N}]^\omega$, and $\gamma \leq \beta$ be a limit ordinal number and let $\eta(\gamma, n)$ be the sequence provided by Proposition 2.6. Assume that A_0 a singleton or the empty set and that $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \rightarrow X$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X starting at A_0 that is c -refined.

If A_0 is a singleton, say $A_0 = \{a_0\}$, let $\Psi : \mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [M]^{<\omega} \rightarrow X$, with $\Psi(A) = \Phi(A)$, be the semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, a_0)} \cap [M]^{<\omega}$ into X starting at A_0 , that is c -refined, given by Remark 9.6. Then, for every $E \in \text{MAX}(\mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [N]^{<\omega})$, we have

$$(114) \quad ((x_{\Psi, A}^{(k)})_{k=1}^{s(\beta, \eta(\gamma, a_0), A)})_{A \in \mathcal{E}_{\beta, \eta(\gamma, a_0)}(A_0 \cup E)} = ((x_{\Phi, A}^{(k)})_{k=1}^{s(\beta, \gamma, A)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)}.$$

If $A_0 = \emptyset$, let $a_0 \in M$, set $A_0 = \{a_0\}$, $N = M \cap [a_0, \infty)$ and $\Psi = \Phi|_{\mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [N]^{<\omega}}$, which is, by Remark 9.6, a semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, a_0)} \cap [N]^{<\omega}$ into X starting at A_0 , that is c -refined. Then, again for every $E \in \text{MAX}(\mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [N]^{<\omega})$,

$$(115) \quad ((x_{\Psi, A}^{(k)})_{k=1}^{s(\beta, \eta(\gamma, a_0), A)})_{A \in \mathcal{E}_{\beta, \eta(\gamma, a_0)}(A_0 \cup E)} = ((x_{\Phi, A}^{(k)})_{k=1}^{s(\beta, \gamma, A)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)}$$

is the step block decomposition of E with respect to Ψ .

Before formulating and proving the missing parts our Main Theorems A and B (see upcoming Theorem 11.6) we present the argument which is the main inductive step.

Let $\gamma < \beta$. For $B \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)}) = \mathcal{S}_\beta[\mathcal{S}_{\beta\gamma}]$ (Proposition 2.6), we let $B_1 < B_2 < \dots < B_d$ the (unique) elements of $\text{MAX}(\mathcal{S}_{\beta\gamma})$ for which $B = \bigcup_{j=1}^d B_j$. We also define $\bar{B} = \{\min(B_j) : j = 1, 2, \dots, d\} \in \text{MAX}(\mathcal{S}_\beta)$ and for $i = 1, \dots, d$, $\bar{B}_j = \{\min(B_j) : j = 1, 2, \dots, i\}$.

If $\emptyset \prec A \preceq B$ we can write A as $A = \bigcup_{i=1}^{j-1} B_i \cup C$, for some $j = 1, 2, \dots, d$, and some $\emptyset \prec C \preceq B_j$, and thus, by Proposition 3.4 $\zeta(\beta(\gamma+1), A) = \zeta(\beta, \bar{B}_j)\zeta(\beta\gamma, C)$. We define

$$(116) \quad \tilde{\mathcal{E}}_{\beta, \gamma+1}(B) = \left\{ \left(\bigcup_{i=1}^j B_i \right) \cup C \in \mathcal{E}_{\beta, \gamma+1}(B) : 1 \leq j < d, l_1(\bar{B}_j) > 0 \text{ and } C \preceq B_{j+1} \right\}.$$

Let $M \in [N]^\omega$, A_0 be a subset of \mathbb{N} that is either empty or a singleton, and $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [M]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [M]^{<\omega}$ into X , starting after A_0 , that is c -refined,

for some $0 < c \leq 1$. Let also $E \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)})(A_0) \cap [M]^{<\omega}$, and put $B = A_0 \cup E$. For $1 \leq j < d$, with $l_1(\bar{B}_j) > 0$, put $M^{(j)} = M \cap (\max(B_j), \infty)$, and $\Phi_E^{(j)} : \mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega} \rightarrow X$ with

$$(117) \quad \Phi_E^{(j)}(C) = \frac{1}{\zeta(\beta, \bar{B}_{j+1})} \Phi((\cup_{1 \leq i \leq j} B_i) \setminus A_0 \cup C).$$

Recall that by Lemma 9.7 $\Phi_E^{(j)}$ is a semi embedding of $\mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}$ into X starting at \emptyset , that is c refined, recall that for $1 \leq j < d$

$$\mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E) = \{A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E) : \text{Cp}_{\beta, \gamma}(A, 1) = \cup_{i \leq j} B_i\},$$

and, moreover, if $l_1(\bar{B}_j) > 0$ define

$$(118) \quad y_{\Phi, E}^{(j)} = \sum_{A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)} \frac{\zeta(\beta(\gamma+1), A)}{\zeta(\beta, \bar{B}_j)} \sum_{k=2}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)}.$$

Remark. 1) By Lemma 9.7, each $\Phi_E^{(j)}$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}$ into X , starting at \emptyset , that is c -refined.

2) We note for later use that $(y_{\Phi, E}^{(j)})_{j=1}^d$ is a sequence in X which satisfies the conditions of the sequence $(x_j)_{j=1}^d$ in Theorem 6.1 with $\alpha = \beta$ and thus, assuming that $\text{Sz}(X) \leq \omega^\beta$, also its conclusion.

3) By the definition of the components of a set A , we conclude (note that for $j = d$ we have $l_1(\bar{B}_d) = 0$)

$$(119) \quad \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E) = \bigcup_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E),$$

which yields

$$(120) \quad \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)} = \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_j) y_{\Phi, E}^{(j)} + \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) x_{\Phi, A}^{(1)}.$$

Lemma 11.2. Let $\gamma < \beta$, $M \in [N]^\omega$, A_0 be either empty or a singleton in \mathbb{N} , and let $\Phi : \mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [M]^{<\omega}$ into X , starting after A_0 , that is c -refined, for some $0 < c \leq 1$. Then, for every $E \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)})(A_0) \cap [M]^{<\omega}$

$$(121) \quad \left\| \Phi(E) - \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)} \right\| < \frac{3}{\min(A_0 \cup E)}.$$

Proof. Recall that for $A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)$, we have $x_A = \sum_{k=1}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)}$ and that $\Phi(E) = \sum_{A \preceq A_0 \cup E} \zeta(\beta(\gamma+1), A) x_A$. Hence,

$$\left\| \Phi(E) - \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma, A)} x_{\Phi, A}^{(k)} \right\| = \left\| \sum_{\substack{A \preceq A_0 \cup E: \\ A \notin \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)}} \zeta(\beta(\gamma+1), A) x_A \right\|.$$

We calculate:

$$\begin{aligned}
(122) \quad & \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)}} \zeta(\beta(\gamma+1), A) \\
&= \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)}} \zeta(\beta(\gamma+1), A) + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j)=0}} \sum_{A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \\
&= \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)}} \zeta(\beta(\gamma+1), A) + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j)=0}} \zeta(\beta, \bar{B}_{j+1}) \sum_{A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)} \zeta(\beta\gamma, A \setminus (\cup_{i \leq j} B_j)) \\
&< \frac{2}{\min(A_0 \cup E)} + \frac{1}{\min(\bar{B})} = \frac{3}{\min(A_0 \cup E)},
\end{aligned}$$

where the last inequality follows from Lemmas 3.7 (v) and 10.5. \square

Lemma 11.3. *Let $\gamma, M \in [N]^\omega$, A_0, Φ , and c be as in the statement of Lemma 11.2. If $E \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega})$ and $B_1 < \dots < B_d$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma})$, with $A_0 \cup E = \cup_{j=1}^d B_j$, and $\bar{B} = \{\min B_j : 1 \leq j \leq d\} \in \text{MAX}(\mathcal{S}_\beta)$, and if $(y_{E, \Phi}^{(j)})_{j=1}^d$ is defined as in (118), then for $j = 1, \dots, d-1$ with $l_1(\bar{B}_j) > 0$, where $\bar{B}_j = \{\min B_i : 1 \leq i \leq j\}$, we have $\|y_{\Phi, E}^{(j)}\| \leq 1$ and $\text{ran}(y_{\Phi, E}^{(j)}) \subset (\max(\bar{B}_j), \max(\bar{B}_{j+2}))$. Put $\max(\bar{B}_{d+1}) = \infty$. Then*

$$\begin{aligned}
(123) \quad & \sum_{A \in \tilde{\mathcal{E}}_{\beta(\gamma+1)}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)} \\
&= \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) x_{\Phi, A}^{(1)} + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_{j+1}) y_{\Phi, E}^{(j)}.
\end{aligned}$$

Proof. Observe that (123) immediately follows from (120) and the fact that for $1 \leq j < d$ with $l_1(\bar{B}_j) > 0$, we have $\zeta(\beta, \bar{B}_j) = \zeta(\beta, \bar{B}_{j+1})$. For $1 \leq j < d$ with $l_1(\bar{B}_j) > 0$ and $A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)$ note that $\cup_{i=1}^j B_i \prec A \preceq \cup_{i=1}^{j+1} B_i$ and

$$\begin{aligned}
(124) \quad & u_A = \sum_{k=2}^{s(\beta, \gamma+1, A)} x_{\Phi, A}^{(k)} \\
&= P_{(\max \text{supp}(x_{C_{\mathcal{P}\beta, \gamma}(A, 1)}), \max \text{supp}(x_A))}(x_A) = P_{(\max \text{supp}(x_{\cup_{i=1}^j B_j}), \max \text{supp}(x_A))} x_A
\end{aligned}$$

i.e. $\|u_A\| \leq 1$ and by Definition 9.5 (ii) we obtain

$$\begin{aligned}
& \max(\bar{B}_j) \leq \max(\cup_{i=1}^j B_i) \leq \max \text{supp}(x_{\cup_{i=1}^j B_i}) < \min \text{supp}(u_A) \text{ and} \\
& \max \text{supp}(u_A) \leq \max \text{supp}(x_A) < \min\{m \in M : m > \max(A)\} \leq \max(\bar{B}_{j+2})
\end{aligned}$$

which yields

$$(125) \quad \text{ran}(u_A) \subset (\max(\bar{B}_j), \max(\bar{B}_{j+2})).$$

Furthermore, we have $\zeta(\beta(\gamma+1), A) = \zeta(\beta, \bar{B}_{j+1})\zeta(\beta\gamma, A \setminus (\cup_{i=1}^j B_i))$, and since $\zeta(\beta, \bar{B}_{j+1}) = \zeta(\beta, \bar{B}_j)$ (by $l_1(\bar{B}_j) > 0$), we obtain

$$(126) \quad \frac{\zeta(\beta(\gamma+1), A)}{\zeta(\beta, \bar{B}_j)} = \zeta(\beta\gamma, A \setminus (\cup_{i=1}^j B_i)).$$

We deduce

$$y_{\Phi, E}^{(j)} = \sum_{C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})} \zeta(\beta\gamma, C) u_{(\cup_{i=1}^j B_i) \cup C}.$$

The above in combination with (125), (126), and the fact that $\|u_A\| \leq 1$ yield that $\|y_{\Phi, E}^{(j)}\| \leq 1$ and $\text{ran}(y_{\Phi, E}^{(j)}) \subset (\max(\bar{B}_j), \max(\bar{B}_{j+2}))$. \square

Lemma 11.4. *Let $\gamma, M \in [N]^\omega$, A_0, Φ , and c be as in the statement of Lemma 11.2. Let $E \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [M]^{<\omega})$, $(B_j)_{j=1}^d, \bar{B}, (\bar{B}_j)_{j=1}^d$ be as in the statement of Lemma 11.3 and let $\Phi_E^{(j)}$, $j = 1, \dots, d-1$ and $M^{(j)} \in [M]^\omega$ be defined as in (117). For $j = 1, \dots, d-1$ with $l_1(\bar{B}_j) > \emptyset$ denote by*

$$\left(\left(z_{\Phi_E^{(j)}, C}^{(k)} \right)_{k=1}^{s(\beta, \gamma', C)} \right)_{C \in \mathcal{E}_{\beta, \gamma'}(B_{j+1})}$$

the block step decomposition of B_{j+1} with respect to $\Phi_E^{(j)}$. Then for $C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})$ we have $s(\beta, \gamma+1, (\cup_{i=1}^j B_i) \cup C) = s(\beta, \gamma, C) + 1$ and

$$z_{\Phi_E^{(j)}, C}^{(k)} = x_{\Phi, (\cup_{i=1}^j B_i) \cup C}^{(k+1)},$$

for $k = 1, \dots, s(\beta, \gamma, C)$.

Proof. Fix $C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})$. By (107b), if we set $A = (\cup_{i=1}^j B_i) \cup C$, then $s(\beta, \gamma+1, A) = s(\beta, \gamma, C) + 1$ and $\text{Cp}_{\beta, \gamma+1}(A, i+1) = \text{Cp}_{\beta, \gamma}(A, i)$ for $i = 1, \dots, s(\beta, \gamma, C)$, and $\text{Cp}_{\beta, \gamma+1}(A, 1) = \cup_{i=1}^j B_i$. Fix $1 \leq k \leq s(\beta, \gamma, C)$. Let $\{z_C : C \in \mathcal{S}_{\beta\gamma} \cap [M^{(j)}]^{<\omega}\}$ be the family generating $\Phi_E^{(j)}$ and $n_0 = \max \text{supp}(x_{\cup_{i=1}^j B_i})$, then by Definition 11.1 and Lemma 9.7 we have

$$x_{\Phi, A}^{(k+1)} = P_{I(A, k+1)}(x_A) \text{ and } z_{\Phi_E^{(j)}, C}^{(k)} = P_{I(C, k)}(z_C) = P_{I(C, k)}(P_{(n_0, \infty)}(x_{\cup_{i \leq j} B_i \cup C})),$$

with (if $k = 1$ replace $\max \text{supp}(z_{\cup_{i=1}^{k-1} \text{Cp}_{\beta, \gamma}(C, i)})$ by n_0)

$$\begin{aligned} I(C, k) &= (\max \text{supp}(z_{\cup_{i=1}^{k-1} \text{Cp}_{\beta, \gamma}(C, i)}), \max \text{supp}(z_{\cup_{i=1}^k \text{Cp}_{\beta, \gamma}(C, i)})) \\ &= (\max \text{supp}(P_{(n_0, \infty)} x_{(\cup_{i=1}^j B_i) \cup (\cup_{i=1}^{k-1} \text{Cp}_{\beta, \gamma}(C, i))}), \max \text{supp}(P_{(n_0, \infty)} x_{(\cup_{i=1}^j B_i) \cup (\cup_{i=1}^k \text{Cp}_{\beta, \gamma}(C, i))})) \\ &= (\max \text{supp}(x_{(\cup_{i=1}^j B_i) \cup (\cup_{i=1}^{k-1} \text{Cp}_{\beta, \gamma}(C, i))}), \max \text{supp}(x_{(\cup_{i=1}^j B_i) \cup (\cup_{i=1}^k \text{Cp}_{\beta, \gamma}(C, i))})) = I(A, k+1), \end{aligned}$$

where we used $n_0 \leq \max \text{supp}(x_{\cup_{i=1}^j B_i})$, which follows from Definition 9.5 (ii). Hence,

$$z_{\Phi_E^{(j)}, C}^{(k)} = P_{I(A, k+1)} P_{(n_0, \infty)}(x_A) = P_{I(A, k+1)}(x_A) = x_{\Phi, A}^{(k+1)}.$$

\square

Proposition 11.5. *Assume that $\text{Sz}(X) \leq \omega^\beta$. Then, for every $1 \geq c > 0$ and $M \in [\mathbb{N}]^\omega$, there exists $N \in [M]^\omega$ with the following property: for every $\gamma \leq \beta$, $L \in [N]^\omega$, and $A_0 \subset N$ that is either empty or a singleton, every semi-embedding $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \rightarrow X$ from $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ into X starting at A_0 , that is c -refined, every $E \in \text{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$ and every (β, γ) -special family of convex combinations $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E), 1 \leq k \leq s(\beta, \gamma, A)\}$ we have*

$$(127) \quad \sum_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)} \zeta(\beta\gamma, A) \sum_{k=1}^{s(\beta, \gamma, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \geq \frac{c}{3},$$

where $(x_{\Phi, A}^{(k)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)}$ is the block step decomposition of E for Φ (Definition 11.1).

Proof. Fix $M \in [\mathbb{N}]^\omega$ and $1 \geq c > 0$. Choose $\varepsilon > 0$ so that $c/2 - \varepsilon > c/3$ and then apply Theorem 6.1 to find $N \in [M]^\omega$ so that (69) is satisfied for that ε and β , and, moreover,

$$(128) \quad \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(N)}\right) \prod_{m \in N} \left(1 - \frac{1}{m}\right) > \frac{c}{3}.$$

We claim that this is the desired set. We shall prove by transfinite induction on γ the following statement: if $\gamma \leq \beta$, $L \in [N]^\omega$, $A_0 \subset L$ that is either empty or a singleton, and $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \rightarrow X$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ into X starting at A_0 , that is c -refined, then for any $E \in \text{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$ and any (β, γ) -special family of convex combinations $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E), 1 \leq k \leq s(\beta, \gamma, A)\}$ we have

$$(129) \quad \sum_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)} \zeta(\beta\gamma, A) \sum_{k=1}^{s(\beta, \gamma, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \geq \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)}\right) \prod_{m \in L} \left(1 - \frac{1}{m}\right).$$

In conjunction with (128), this will yield the desired result. Let $\gamma = 1$, $L \subset N$, A_0 be a subset of L that is either empty or a singleton, and $\Phi : \mathcal{S}_\beta(A_0) \cap [L]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_\beta \cap [L]^{<\omega}$ into X starting at A_0 that is c -refined. Let $E \in \text{MAX}(\mathcal{S}_\beta(A_0) \cap [N]^{<\omega})$.

By (107a), for each $A \in \mathcal{E}_{\beta, 1}(A_0 \cup E)$ we obtain that the block step decomposition of x_A is just $(x_{\Phi, A}^{(1)}) = (x_A)$, and hence if $\{r(A, k) : A \in \mathcal{E}_{\beta, 1}(A_0 \cup E), 1 \leq k \leq s(\beta, 1, A)\}$ is a $(\beta, 1)$ -special family of convex combinations then $r(A, 1) = 1$. Hence,

$$\begin{aligned} \sum_{A \in \mathcal{E}_{\beta, 1}(A_0 \cup E)} \zeta(\beta, A) \sum_{k=1}^{s(\beta, 1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| &= \sum_{A \in \mathcal{E}_{\beta, 1}(A_0 \cup E)} \zeta(\beta, A) \|x_A\| \\ &\geq \left\| \sum_{A \preceq A_0 \cup E} \zeta(\beta, A) x_A \right\| - \sum_{\substack{A \preceq A_0 \cup E \\ A \notin \mathcal{E}_{\beta, 1}(A_0 \cup E)}} \zeta(\beta, A) \\ &\geq \|\Phi(E)\| - \frac{2}{\min(A_0 \cup E)} \quad (\text{by (112)}) \\ &\geq \frac{c}{2} - \frac{c}{2} \zeta(\beta, A_0) - \frac{2}{\min(A_0 \cup B)} \quad (\text{by 9.5 (iv)}) \\ &> \frac{c}{2} - \frac{3}{\min(L)} \quad (\text{by (33)}). \end{aligned}$$

To verify the induction step, let first $\gamma < \beta$ be an ordinal number for which the conclusion holds. Let $L \subset N$, A_0 be a subset of L that is either empty or a singleton, and $\Phi :$

$\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [L]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta(\gamma+1)} \cap [L]^{<\omega}$ into X starting at A_0 that is c -refined. Let $E \in \text{MAX}(\mathcal{S}_{\beta(\gamma+1)}(A_0) \cap [L]^{<\omega})$ and $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E), 1 \leq k \leq s(\beta, \gamma+1, A)\}$ be a $(\beta, \gamma+1)$ -special family of convex combinations. Let $A_0 \cup E = \bigcup_{j=1}^d B_j$, where $B_1 < \dots < B_d$ are in $\text{MAX}(\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega})$ and $\bar{B} = \min\{B_j : 1 \leq j \leq d\}$ is in $\text{MAX}(\mathcal{S}_\beta)$. By Lemma 11.3 and the choice of the set N , we obtain

$$(130) \quad \left\| \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_{j+1}) y_{\Phi, E}^{(j)} \right\| < \varepsilon.$$

Combining first Lemmas 11.2 and 11.3, then applying Definition 9.5 (iv), (33), and finally using (130) we deduce

$$(131) \quad \begin{aligned} & \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \|x_{\Phi, A}^{(1)}\| \\ & \geq \left\| \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) x_{\Phi, A}^{(1)} \right\| \\ & \geq \|\Phi(E)\| - \left\| \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_{j+1}) y_{\Phi, E}^{(j)} \right\| - \frac{3}{\min(A_0 \cup E)} \\ & \geq \frac{c}{2} - \frac{c}{2} \zeta(\beta(\gamma+1), A_0) - \varepsilon - \frac{3}{\min(A_0 \cup E)} \geq \frac{c}{2} - \varepsilon - \frac{4}{\min(L)}. \end{aligned}$$

We distinguish two cases. If $r(A, 1) = 1$, for all $A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)$, then

$$\begin{aligned} & \sum_{A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \\ & = \sum_{A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \|x_{\Phi, A}^{(1)}\| \geq \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \|x_{\Phi, A}^{(1)}\|. \end{aligned}$$

By (131), the result follows in that case.

Otherwise we have $r_1 = r(A, 1) < 1$, for all $A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)$. For $1 \leq j < d$, with $l_1(\bar{B}_j) > 0$, define $L^{(j)} = L \cap (\max(A_0 \cup (\bigcup_{1 \leq i \leq j} B_i)), \infty)$, and $\Phi_E^{(j)} : \mathcal{S}_{\beta\gamma} \cap [L^{(j)}]^{<\omega} \rightarrow X$ as in (117). By Lemma 9.7, each $\Phi_E^{(j)}$ is a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [L^{(j)}]^{<\omega}$ into X , starting at \emptyset , that is c -refined.

By Lemma 10.9, the family $\{r^{(j)}(C, k) : C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})\}$, with

$$r^{(j)}(C, k) = \frac{1}{1 - r(A^{(1)}, 1)} r\left(\left(\bigcup_{i=1}^j B_i\right) \cup C, k+1\right)$$

for $k = 1, \dots, s(\beta, \gamma, C) = s(\beta, \gamma, (\bigcup_{i=1}^j B_i) \cup C) - 1$ and some $A^{(1)} \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)$, is a (β, γ) -special family of convex combinations. Hence, by the inductive assumption applied to the map $\Phi_E^{(j)}$, and Lemma 11.4 we deduce

$$(132) \quad \sum_{A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=2}^{s(\beta, \gamma+1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\|$$

$$\begin{aligned}
&= \zeta(\beta, \bar{B}_j)(1 - r(A^{(1)}, 1)) \sum_{C \in \mathcal{E}_{\beta, \gamma}(B_{j+1})} \zeta(\beta, \gamma, C) \sum_{k=1}^{s(\beta, \gamma, C)} r^{(j)}(C, k) \|z_{\Phi_E^{(j)}, C}^{(k)}\| \\
&\geq \zeta(\beta, \bar{B}_j)(1 - r(A^{(1)}, 1)) \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L^{(j)})} \right) \prod_{m \in L^{(j)}} \left(1 - \frac{3}{m} \right).
\end{aligned}$$

We combine (131) with (132):

$$\begin{aligned}
&\sum_{A \in \mathcal{E}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \\
&\geq \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=1}^{s(\beta, \gamma+1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \\
&= \sum_{A \in \tilde{\mathcal{E}}_{\beta, \gamma+1}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) r(A, 1) \|x_{\Phi, A}^{(1)}\| \\
&\quad + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \sum_{A \in \mathcal{E}_{\beta, \gamma+1}^{(j)}(A_0 \cup E)} \zeta(\beta(\gamma+1), A) \sum_{k=2}^{s(\beta, \gamma+1, A)} r(A, k) \|x_{\Phi, A}^{(k)}\| \\
&\geq r(A^{(1)}, 1) \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \\
&\quad + \sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_j)(1 - r(A^{(1)}, 1)) \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \prod_{m \in L^{(j)}} \left(1 - \frac{1}{m} \right) \\
&\geq \left(\sum_{\substack{1 \leq j < d \\ l_1(\bar{B}_j) > 0}} \zeta(\beta, \bar{B}_j) \right) \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \prod_{m \in L^{(1)}} \left(1 - \frac{1}{m} \right) \\
&\geq \left(1 - \frac{1}{\min(\bar{B})} \right) \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \prod_{m \in L^{(1)}} \left(1 - \frac{1}{m} \right) \text{ (by Lemma 3.7 (v))} \\
&\geq \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \prod_{m \in L} \left(1 - \frac{1}{m} \right).
\end{aligned}$$

Assume now that $\gamma \leq \beta$ is a limit ordinal number and that the claim holds for all $\gamma' < \gamma$. Let $L \in [N]^\omega$, A_0 be a subset of L that is either empty or a singleton, and $\Phi : \mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega} \rightarrow X$ be a semi-embedding of $\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega}$ into X starting at A_0 that is c -refined. We distinguish between two cases, namely whether A_0 is a singleton or whether it is empty. In the first case, $A_0 = \{a_0\}$, for some $a_0 \in L$. By Remark 9.6, the map Ψ with $\Psi = \Phi$ can be seen as a semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, a_0)} \cap [L]^{<\omega}$ into X starting at A_0 , that is c -refined. If $E \in \text{MAX}(\mathcal{S}_{\beta\gamma}(A_0) \cap [L]^{<\omega})$ then $E \in \text{MAX}(\mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [L]^{<\omega})$ and by (114) we have

$$((x_{\Psi, A}^{(k)})_{k=1}^{s(\beta, \eta(\gamma, a_0), A)})_{A \in \mathcal{E}_{\beta, \eta(\gamma, a_0)}(A_0 \cup E)} = ((x_{\Phi, A}^{(k)})_{k=1}^{s(\beta, \gamma, A)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)},$$

whereas if $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E), 1 \leq k \leq s(\beta, \gamma, A)\}$ is a (β, γ) -special family of convex combinations, by the remark following Definition 10.7, it is a $(\beta, \eta(\gamma, a_0))$ -special family of

convex combinations as well. Applying the inductive assumption for $\eta(\gamma, a_0) < \gamma$ yields

$$\begin{aligned}
 (133) \quad & \sum_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)} \zeta(\beta\gamma, A) \sum_{k=1}^{s(\beta, \gamma, A)} r(A, k) \left\| x_{\Phi, A}^{(k)} \right\| \\
 &= \sum_{A \in \mathcal{E}_{\beta, \eta(\gamma, a_0)}(A_0 \cup E)} \zeta(\beta\eta(\gamma, a_0), A) \sum_{k=1}^{s(\beta, \eta(\gamma, a_0), A)} r(A, k) \left\| x_{\Psi, A}^{(k)} \right\| \\
 &\geq \left(\frac{c}{2} - \varepsilon - \frac{4}{\min(L)} \right) \prod_{m \in L} \left(1 - \frac{1}{m} \right).
 \end{aligned}$$

In the second case, A_0 is empty. Let $B \in \text{MAX}(\mathcal{S}_{\beta\gamma} \cap [L]^{<\omega})$ and set $a_0 = \min(B)$. By Remark 9.6, if $L' = L \cap [a_0, \infty)$, then the map $\Psi : \mathcal{S}_{\beta\eta(\gamma, a_0)}(A_0) \cap [L'] \rightarrow X$ with $\Psi(A) = \Phi(A_0 \cup A)$, is a semi-embedding of $\mathcal{S}_{\beta\eta(\gamma, a_0)} \cap [L']^{<\omega}$ into X starting at A_0 that is c -refined. By (115) we obtain

$$((x_{\Psi, A}^{(k)})_{k=1}^{s(\beta, \eta(\gamma, a_0), A)})_{A \in \mathcal{E}_{\beta, \eta(\gamma, a_0)}(B)} = ((x_{\Phi, A}^{(k)})_{k=1}^{s(\beta, \gamma, A)})_{A \in \mathcal{E}_{\beta, \gamma}(A_0 \cup E)}$$

whereas if $\{r(A, k) : A \in \mathcal{E}_{\beta, \gamma}(B), 1 \leq k \leq s(\beta, \gamma, A)\}$ is (β, γ) -special family of convex combinations, by the Remark following Definition 10.8, it is a $(\beta, \eta(\gamma, a_0))$ -special family of convex combinations as well. The result follows in the same manner as in (133). \square

Theorem 11.6. *Assume that X is a reflexive and separable Banach space, with the property that $\text{Sz}(X) \leq \omega^\beta$ and $\text{Sz}(X^*) < \beta$. Then for no $L \in [\mathbb{N}]^\omega$, does there exists a semi-embedding of $\mathcal{S}_{\beta^2} \cap [L]^{<\omega}$ into X , starting at \emptyset .*

Proof. By the main Theorem of [33] we can embed X into a reflexive space Z with basis, so that $\text{Sz}(Z) = \text{Sz}(X)$, and $\text{Sz}(Z^*) = \text{Sz}(X^*)$. Thus we may assume that X has a basis, which must be shrinking and boundedly complete, since X is reflexive. By re-norming X , we may assume that the bases of X and X^* are bimonotone. Choose α_0 with $\text{Sz}(X^*) \leq \omega^{\alpha_0} < \beta$ (this is possible due to the form of β). Note that

$$(134) \quad \text{CB}(\mathcal{S}_{\alpha_0}) = \omega^{\alpha_0} + 1 < \beta + 1 = \text{CB}(\mathcal{F}_{\beta, \beta}).$$

Towards a contradiction, assume that there exists $L \in [\mathbb{N}]^\omega$ and a semi-embedding Ψ of $\mathcal{S}_{\beta^2} \cap [L]^{<\omega}$ into X , starting at \emptyset . By Lemma 9.8 there exist $1 \geq c > 0$, $M \in [L]^\omega$, and a semi-embedding Φ of $\mathcal{S}_{\beta^2} \cap [M]^{<\omega}$ into X , starting at \emptyset , that is c -refined. Applying Proposition 11.5, we may pass to a further subset of M , again denoted by M , so that (127) holds.

Fix $0 < \varepsilon < c/3$ and apply Theorem 6.1 to the space X^* and the ordinal number α_0 to find a further subset of M , which we again denote by M , so that (69) is satisfied.

By Propositions 2.2 and 2.13 we may pass to a subset of M , again denoted by M , so that $\mathcal{S}_{\alpha_0} \cap [M]^{<\omega} \subset \mathcal{F}_{\beta, \beta}$. By Lemma 10.2 we obtain that for any $B \in \text{MAX}(\mathcal{S}_{\beta^2} \cap [M]^{<\omega})$ and $A \in \mathcal{E}_{\beta, \beta}(B)$, there exists $\tilde{A} \in \text{MAX}(\mathcal{S}_{\alpha_0})$ with

$$(135) \quad \tilde{A} \preceq \{\min(\text{Cp}_{\beta, \beta}(A, k)) : 1 \leq k \leq s(\beta, \beta, A)\}.$$

Choose $B \in \text{MAX}(\mathcal{S}_{\beta^2} \cap [M]^{<\omega})$. We will define a (β, β) -special family of convex combinations $\{r(A, k) : A \in \mathcal{E}_{\beta, \beta}(B), 1 \leq k \leq s(\beta, \beta, A)\}$. For $A \in \mathcal{E}_{\beta, \beta}(B)$ let $\tilde{A} = \{a_1^A, \dots, a_{d_A}^A\} \in \text{MAX}(\mathcal{S}_{\alpha_0})$ be as in (135). For $1 \leq k \leq s(\beta, \beta, A)$ set

$$(136) \quad r(A, k) = \begin{cases} \zeta(\alpha_0, \tilde{A}_k) & \text{if } k \leq \#\tilde{A} \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{A}_k = \{a_1^A, \dots, a_k^A\}$ for $1 \leq k \leq \#\tilde{A}$. We will show that this family satisfies Definition 10.8 (i) and (ii). The first assertion is straightforward, to see the second one let $A^{(1)}, A^{(2)} \in \mathcal{E}_{\beta,\beta}(B)$, so that if $(D_k(A^{(1)}))_{k=1}^{s(\beta,\beta,A)}$ and $(D_k(A^{(2)}))_{k=1}^{s(\beta,\beta,A')}$ are the maximal chains of $\mathcal{A}_{\beta,\beta}(B)$ provided by Lemma 10.2, then for some k we have $D_k(A^{(1)}) = D_k(A^{(2)})$. By Lemmas 10.6 and 10.7 we obtain $\min(\text{Cp}_{\beta,\beta}(A^{(1)}, m)) = \min(\text{Cp}_{\beta,\beta}(A^{(2)}, m))$ for $m = 1, \dots, k$, which implies $\tilde{A}_m^{(1)} = \tilde{A}_m^{(2)}$ for $m = 1, \dots, \min\{k, \#\tilde{A}^{(1)}\}$. By (136) it easily follows that $r(A^{(1)}, k) = r(A^{(2)}, k)$. Since (127) is satisfied, we obtain

$$(137) \quad \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) \|x_{\Phi,A}^{(k)}\| \geq \frac{c}{3}.$$

For each $A \in \mathcal{E}_{\beta,\beta}(B)$ and $k = 1, \dots, d_A$ choose $f_A^{(k)}$ in S_{X^*} , with $f_A^{(k)}(x_A^{(k)}) = \|x_A^{(k)}\|$ and $\text{ran}(f_A^{(k)}) \subset \text{ran}(x_{\Phi,A}^{(k)})$

$$\begin{aligned} &\subset \left(\max \text{supp}(x_{\cup_{i=1}^{k-1} \text{Cp}_{\beta,\beta}(A,i)}), \max \text{supp}(x_{\cup_{i=1}^k \text{Cp}_{\beta,\beta}(A,i)}) \right) \\ &\subset \left(\min(\text{Cp}_{\beta,\beta}(A, k-1)), \min(\text{Cp}_{\beta,\beta}(A, k+1)) \right) = (\max(\tilde{A}_{k-1}), \max(\tilde{A}_{k+1})), \end{aligned}$$

where the third inclusion follows from Definition 9.5 (ii). As (69) is satisfied, we obtain that for all $A \in \mathcal{E}_{\beta,\beta}(B)$

$$(138) \quad \left\| \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)} \right\| < \varepsilon.$$

We finally calculate

$$\begin{aligned} \frac{c}{3} &\leq \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) \|x_{\Phi,A}^{(k)}\| \quad (\text{by (137)}) \\ &= \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}(x_{\Phi,A}^{(k)}) \quad (\text{by choice of } f_A^{(k)}) \\ &= \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)} \left(\sum_{m=1}^{s(\beta,\beta,A)} x_{\Phi,A}^{(m)} \right) \quad (\text{since } \text{ran}(f_A^{(k)}) \subset \text{ran}(x_{\Phi,A}^{(k)})) \\ &= \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)}(x_A) \quad (\text{by (113)}) \\ &\leq \sum_{A \in \mathcal{E}_{\beta,\beta}(B)} \zeta(\beta^2, A) \left\| \sum_{k=1}^{d_A} \zeta(\alpha_0, \tilde{A}_k) f_A^{(k)} \right\| < \varepsilon \quad (\text{by (138)}). \end{aligned}$$

This contradiction completes the proof. \square

Before proving Corollary 1.2 we will need the following observation.

Proposition 11.7. *Let X be a Banach space, $\alpha < \omega_1$ and L be an infinite subset of the natural numbers so that there exist numbers $0 < c < C$ and a map $\Phi : \mathcal{S}_\alpha \cap [L]^{<\omega} \rightarrow X$ that*

is a c -lower- $d_{\alpha,\infty}$ and C -upper- $d_{\alpha,1}$ embedding. Then for every $\beta < \alpha$ there exists $n \in \mathbb{N}$ and a map $\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n, \infty)]^{<\omega} \rightarrow X$ that is a c -lower- $d_{\beta,\infty}$ and C -upper- $d_{\beta,1}$ embedding.

Before proving Proposition 11.7 we need some preliminary observation.

Lemma 11.8. *Let α be an ordinal number and $A \subset [0, \alpha]$ satisfying:*

- (i) $\alpha \in A$ and
- (ii) if $\beta \in A$ and $\gamma < \beta$, then there is $\gamma \leq \eta < \beta$ with $\eta \in A$.

Then $A = [0, \alpha]$.

Proof. If the conclusion is false, let α_0 be the minimum ordinal for which there exists $A_0 \subsetneq [0, \alpha_0]$ satisfying (i) and (ii). Fix an arbitrary $\gamma < \alpha_0$. By the properties of A_0 , there exists $\gamma \leq \alpha < \alpha_0$ with $\alpha \in A_0$. If we set $A = A_0 \cap [0, \alpha]$, then it easily follows that A and α satisfy (i) and (ii). By the minimality of α_0 , we conclude $A = [0, \alpha]$ and hence, since $\gamma \in [0, \alpha] = A$ and $A \subset A_0$, we have $\gamma \in A_0$. Since γ was chosen arbitrarily we conclude $[0, \alpha_0] = A_0$, a contradiction that completes the proof. \square

Lemma 11.9. *Let $\alpha < \omega_1$ be a limit ordinal number. Then there exists a sequence of successor ordinal numbers $(\mu(\alpha, n))_n$ satisfying:*

- (i) $\mu(\alpha, n) < \alpha$ for all $n \in \mathbb{N}$ and $\lim_n \mu(\alpha, n) = \alpha$,
- (ii) $\mathcal{S}_\alpha = \{A \in [\mathbb{N}]^{<\omega} : A \in \mathcal{S}_{\mu(\alpha, \min(A))}\} \cup \{\emptyset\}$ and $\mathcal{S}_{\mu(\alpha, n)} \cap [[n, \infty)]^{<\omega} \subset \mathcal{S}_\alpha$ for all $n \in \mathbb{N}$, and
- (iii) for $A \in \mathcal{S}_\alpha \setminus \{\emptyset\}$, $z_{(\alpha, A)} = z_{(\mu(\alpha, \min(A)), A)}$.

Proof. We define $(\mu(\alpha, n))_n$ by transfinite recursion on the set of countable limit ordinal numbers. For $\alpha = \omega$ we set $(\mu(\omega, n))_n = (\lambda(\omega, n))_n$. If α is a limit ordinal so that for all $\alpha' < \alpha$ the corresponding sequence has been defined, set for each $n \in \mathbb{N}$

$$\mu(\alpha, n) = \begin{cases} \lambda(\alpha, n) & \text{if } \lambda(\alpha, n) \text{ is a successor ordinal number, or} \\ \mu(\lambda(\alpha, n), n) & \text{otherwise.} \end{cases}$$

The fact that (ii), (iii) and the first part of (i) hold is proved easily by transfinite induction using (23) in Corollary 2.7 and the definition of repeated averages. To show that $\lim_n \mu(\alpha, n) = \alpha$, we will show that for arbitrary $L \in [\mathbb{N}]^\omega$ we have $\sup_{n \in L} \mu(\alpha, n) = \alpha$. Fix $L \in [\mathbb{N}]^\omega$ and $\beta < \alpha$. Then, since $\text{CB}(\mathcal{S}_\alpha \cap [L]^{<\omega}) = \omega^\alpha + 1 > \omega^\beta + 1$, we have $\emptyset \in (\mathcal{S}_\alpha \cap [L]^{<\omega})^{(\omega^\beta + 1)}$ and hence there exists $n \in L$ with $\{n\} \in (\mathcal{S}_\alpha \cap [L]^{<\omega})^{(\omega^\beta)}$. Using (4) we obtain

$$\emptyset \in (\mathcal{S}_\alpha \cap [L]^{<\omega})^{(\omega^\beta)}(\{n\}) \subset \mathcal{S}_\alpha^{(\omega^\beta)}(\{n\}) = (\mathcal{S}_\alpha(\{n\}))^{(\omega^\beta)} = (\mathcal{S}_{\mu(\alpha, n)}(\{n\}))^{(\omega^\beta)} = \mathcal{S}_{\mu(\alpha, n)}^{(\omega^\beta)}(\{n\}),$$

which implies $\{n\} \in \mathcal{S}_{\mu(\alpha, n)}^{(\omega^\beta)}$, i.e. $\text{CB}(\mathcal{S}_{\mu(\alpha, n)}) \geq \omega^\beta + 1$ which yields $\mu(\alpha, n) \geq \beta$. \square

Remark. It is uncertain whether $\mathcal{S}_{\mu(\alpha, n)} \subset \mathcal{S}_{\mu(\alpha, n+1)}$, it is even uncertain whether $\mu(\alpha, n) \leq \mu(\alpha, n+1)$. It is true however that for $2 \leq n \leq m$ we have $\mu(\alpha, n) \leq \mu(\alpha, m) + 1$.

Proof of Proposition 11.7. We shall first treat two very specific cases. In the first case, $\alpha = \beta + 1$. Fix $n_0 \geq 2$ with $n_0 \in L$ and $B_0 \in \text{MAX}(\mathcal{S}_\beta \cap [L]^{<\omega})$ with $\min(B_0) = n_0$. Define $n = \max(B_0)$ and $\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n, \infty)]^{<\omega} \rightarrow X$ with $\Phi_\beta(A) = n_0 \Phi(B_0 \cup A)$. Then Φ_β is the desired embedding.

In the second case, α is a limit ordinal and for some $n_0 \geq 2$ with $n_0 \in L$ we have $\beta + 1 = \mu(\alpha, n_0)$. Fix $B_0 \in \text{MAX}(\mathcal{S}_\beta \cap [L]^{<\omega})$ with $\min(B_0) = n_0$, define $n = \max(B_0)$ and

$\Phi_\beta : \mathcal{S}_\beta \cap [L \cap (n, \infty)]^{<\omega} \rightarrow X$ with $\Phi_\beta(A) = n_0 \Phi(B_0 \cup A)$. Then, using the properties of $(\mu(\alpha, k))_k$, it can be seen that Φ_β is well defined and it is the desired embedding.

In the general case, define A to be the set of all $\beta \leq \alpha$ for which such an n and Φ_β exist. Since $\alpha \in A$, it remains to show that A satisfies (ii) of Lemma 11.8. Indeed, fix $\beta \in A$ and $\gamma < \beta$. If $\beta = \eta + 1$, then by the first case we can deduce that $\eta \in A$ and $\gamma \leq \eta < \beta$. Otherwise, β is a limit ordinal. Let Φ_β and n_β witness the fact that $\beta \in A$ and by Lemma 11.9 (i) we may choose $n \in L$ with $n > n_\beta$ so that $\mu(\beta, n) > \gamma + 1$. If η is the predecessor of $\mu(\beta, n)$, then by the second case we deduce that $\eta \in A$ and $\gamma \leq \eta < \beta$. \square

Proof of Corollary 1.2. We first recall a result by Causey [5, Theorem 6.2] which says that for a countable ordinal ξ it follows that $\gamma = \omega^\xi$ is the Szlenk index of some separable Banach space X , if and only if ξ is not of the form $\xi = \omega^\eta$, with η being a limit ordinal. Since $\alpha = \omega^{\omega^{\omega^\alpha}}$, α cannot be the Szlenk index of some separable Banach space.

“(a) \Rightarrow (b)” From (a) and Causey’s result we have $\text{Sz}(X) < \omega^\alpha$ and $\text{Sz}(X^*) < \omega^\alpha$, and thus there exists a $\xi < \omega_1$ with $\beta = \omega^{\omega^\xi} \leq \omega^{\omega^{\xi+1}} < \alpha$, so that $\text{Sz}(X) < \omega^\beta$ and $\text{Sz}(X) < \beta$. It follows therefore from Theorem 11.6 that for no $L \in [\mathbb{N}]^\omega$ there exist numbers $0 < c < C$ and a map $\Phi : \mathcal{S}_{\beta^2} \cap [L]^{<\omega} \rightarrow X$ that is a c -lower- $d_{\beta^2, \infty}$ and C -upper- $d_{\beta^2, 1}$ embedding. Since $\beta^2 \leq \omega^{\omega^{\xi+1}} < \alpha$ we conclude our claim from Proposition 11.7

“(b) \Rightarrow (a)” follows from Theorems 8.1 and 8.3. \square

In order to proof Corollary 1.3 recall that every separable Banach space is isometrical equivalent to a subspace of $C[0, 1]$, the space of continuous functions on $[0, 1]$. The set \mathcal{SB} of all closed subspaces of $C[0, 1]$ is given the *Effros-Borel structure*, which is the σ -algebra generated by the sets $\{F \in \mathcal{SB} : F \cap U \neq \emptyset\}$, where U ranges over all open subsets of $C[0, 1]$.

Proof of Corollary 1.2. By [26, Theorem D] the set

$$\mathcal{C}_\alpha = \{X \in \mathcal{SB} : X \text{ reflexive, and } \max(\text{Sz}(X), \text{Sz}(X^*)) \leq \alpha\}$$

is analytic. So it is left to show that its complement is also analytic. Since by Corollary 1.3 $\mathcal{SB} \setminus \mathcal{C}_\alpha = \{X \in \mathcal{SB} : X \text{ not reflexive}\} \cup \{X \in \mathcal{SB} : X \text{ reflexive and } \max(\text{Sz}(X), \text{Sz}(X^*)) > \alpha\}$
 $= \{X \in \mathcal{SB} : X \text{ not reflexive}\} \cup \{X \in \mathcal{SB} : (\mathcal{S}_\alpha, d_{1, \alpha}) \text{ bi-Lipschitzly embeds into } X\}$

and since by [10, Corollary 3.3] the set of reflexive spaces in \mathcal{SB} is co-analytic, we deduce our claim from the following Lemma. This Lemma seems to be wellknown, but for completeness we include a short proof. \square

Lemma 11.10. *Let (M, d) be a separable metric space. Then*

$$\mathcal{A}(M) = \{X \in \mathcal{SB} : (M, d) \text{ bi-Lipschitzly embeds into } X\}$$

is analytic in \mathcal{SB} .

Proof. After passing to a countable dense subset of M we can assume that M is countable and we enumerate M into $\{m_n : n \in \mathbb{N}\}$. The set $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, i.e., the set of all sequences in $\mathbb{N}^\mathbb{N}$ is a Polish space with respect to the product topology. We write an element $s \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ as $s = (s_n : n \in \mathbb{N})$ with $s_n = (s_{n,j})_{j=1}^\infty \in \mathbb{N}^\mathbb{N}$. By the Kuratowski-Ryll-Nardzewski Selection Theorem [17, Theorem 12.13] there are Borel maps $d_n : \mathcal{SB} \rightarrow C[0, 1]$, $n \in \mathbb{N}$, so that for every $X \in \mathcal{SB}$, the sequence $(d_n(X))$ is in X and dense in X . Put $D(X) = \{d_n(X) : n \in \mathbb{N}\}$, for $X \in \mathcal{SB}$. We first note that there is a bi-Lipschitz map $\Psi : M \rightarrow X$ if and only if

$$\exists C > 1 \forall n \in \mathbb{N} \exists \Psi_n : M \rightarrow D(X), \text{ so that } (\Psi_n(m)) \text{ is Cauchy in } X \text{ for all } m \in M \text{ and}$$

$$\frac{1}{C}d(m, m') < \lim_{n \rightarrow \infty} \|\Psi_n(m) - \Psi_n(m')\| < Cd(m, m') \text{ for all } m, m' \in M.$$

Since the set

$$\left\{ (s, X) \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times \mathcal{SB} : \begin{array}{l} \forall k \in \mathbb{N} \forall l \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n_1, n_2 \geq n_0 \quad \|d_{s_{n_1}, k}(X) - d_{s_{n_2}, k}(X)\| < \frac{1}{l} \\ \exists C \in \mathbb{N} \forall k, l \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \\ \frac{1}{C}d(m_k, m_l) < \|d_{s_n, k}(X) - d_{s_n, l}(X)\| < Cd(m_k, m_l) \end{array} \right\}$$

is Borel in the product space $\mathcal{SB} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, its projection on the second coordinate is analytic. On the other hand the image of this projection consists exactly of the spaces $X \in \mathcal{SB}$ for which there is a bi-Lipschitz map $\Psi : M \rightarrow X$ (consider for each $s \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, $n \in \mathbb{N}$ and $X \in \mathcal{SB}$, the map $\Psi_n : M \ni m_k \mapsto d_{s_n, k}(X) \in X$). \square

12. FINAL COMMENTS AND OPEN QUESTIONS

The proof of Theorem A, yields the following equivalences. “(a) \Rightarrow (b)” follows from Proposition 7.2, “(d) \Rightarrow (a)” from Theorem 11.6, while “(b) \Rightarrow (c) \Rightarrow (d)” is trivial.

Corollary 12.1. *For a separable Banach space X the following statements are equivalent*

- a) X is not reflexive,
- b) For all $\alpha < \omega_1$ there exists for some numbers $0 < c < C$, a c -lower $d_{\infty, \alpha}$, C -upper $d_{1, \alpha}$ embedding of \mathcal{S}_α into X .
- c) For all $\alpha < \omega_1$ there is a map $\Psi_\alpha : \mathcal{S}_\alpha \rightarrow X$ and some $0 < c \leq 1$, so that for all $A, B, C \in \mathcal{S}_\alpha$, with the property that $A \succeq C$, $B \succeq C$ and $A \setminus C < B \setminus C$

$$cd_{1, \alpha}(A, B) \leq \|\Psi(A) - \Psi(B)\| \leq d_{1, \alpha}(A, B).$$

- d) For all $\alpha < \omega_1$, there is an $L \in [\mathbb{N}]^{<\omega}$ and a semi embedding $\Psi_\alpha : \mathcal{S}_\alpha \cap [L]^{<\omega} \rightarrow X$.

As mentioned before we can consider for $\alpha < \omega_1$ and $A \in \mathcal{S}_\alpha$ the vector z_A to be an element in ℓ_1^+ , with $\|x\|_{\ell_1} \leq 1$. We define

$$T_\alpha = \{(A, B) \in \mathcal{S}_\alpha \times \mathcal{S}_\alpha : \exists C \preceq A \text{ and } C \preceq B, \text{ with } A \setminus C < B \setminus C\}.$$

We note that $d_{1, \alpha}(A, B) = \|z_A - z_B\|_1$, for $(A, B) \in T_\alpha$. Using this notation we deduce the following sharpening of [30, Theorem 3.1] from Corollary 12.1

Corollary 12.2. *Let X be a separable Banach space. Then the following are equivalent:*

- a) X is not reflexive.
- b) For all $\alpha < \omega_1$ there is a map $\Psi_\alpha : \mathcal{S}_\alpha \rightarrow X$ and some $0 < c \leq 1$, so that

$$cd_{1, \alpha}(A, B) \leq \|\Psi_\alpha(A) - \Psi_\alpha(B)\| \leq d_{1, \alpha}(A, B), \text{ whenever } (A, B) \in T_\alpha.$$

Next we would like to show that the James space J (c.f. [14, Definition 4.43]) has the property that $\text{Sz}(J) = \text{Sz}(J^*) = \omega$, and thus, since J is not reflexive, that Theorem C becomes false without the assumption that X is reflexive.

If X is a Banach space with an FDD and $1 < p \leq \infty$. The space X is said to satisfy an upper ℓ_p -estimate, if there is a constant C so that for every block sequence $(x_k)_{k=1}^n$ in X we have $C\|\sum_{k=1}^n x_k\| \leq (\sum_{k=1}^n \|x_k\|^p)^{1/p}$. A lower ℓ_p -estimate is defined in the obvious way. If X is a Banach space with an FDD that satisfies an upper ℓ_p -estimate, note that by a standard argument, the FDD of X has to be shrinking and hence X^* is separable. Therefore the assumption that X has an FDD with an upper ℓ_p -estimate implies by [25, Theorem 3.4] that $\text{Sz}(X) = \omega$.

Example 12.3. Let J denote James space, the above implies that $\text{Sz}(J) = \text{Sz}(J^*) = \omega$. To see that, we recall that J has two bases $(d_i)_i$ and $(e_i)_i$, the first one of which is shrinking and satisfies an upper ℓ_2 -estimate, whereas the second one is boundedly complete and satisfies a lower ℓ_2 -estimate. The first fact yields that $\text{Sz}(J) \leq \omega$, whereas the second fact and a duality argument yields that the space X spanned by the basis $(e_i^*)_i$ satisfies an upper ℓ_2 -estimate, i.e. $\text{Sz}(X) \leq \omega$. It is well known that X is of co-dimension one in J^* and that J^* is isomorphic to its hyperplanes and therefore $\text{Sz}(J^*) \leq \omega$.

This Example shows that Theorem C cannot be true if we omit the requirement that X is reflexive. Nevertheless the following variation of Corollary 1.2 holds.

Corollary 12.4. *Assume that $\alpha < \omega_1$ is an ordinal, for which $\alpha = \omega^\alpha$. Then the following statements are equivalent for a separable Banach space X .*

- a) X is reflexive and $\max(\text{Sz}(X), \text{Sz}(X^*)) \leq \alpha$,
- b) There is no map $\Psi : \mathcal{S}_\alpha \rightarrow X$ and $0 < c \leq 1$ so that for all $A, B, C \in \mathcal{S}_\alpha$ with the property that $C \preceq A$, $C \preceq B$, and $A \setminus C < B \setminus C$, we have

$$cd_{1,\alpha}(A, B) \leq \|\Psi(A) - \Psi(B)\| \leq d_{1,\alpha}(A, B).$$

Proof. Let $\Psi : \mathcal{S}_\alpha \rightarrow X$ satisfying the condition stated in (b) for some $c > 0$. Then $\tilde{\Psi} = (\Psi - \Psi(\emptyset))/2$ also has this property for $c/2$, maps \mathcal{S}_α into B_X and $\tilde{\Psi}(\emptyset) = 0$. “(a) \Rightarrow (b)” follows from Theorem 11.6, the in the proof of Corollary 1.2 cited result from [5], and the same argument involving Proposition 11.7 in the proof of Corollary 1.2. “(b) \Rightarrow (a)” follows from Proposition 7.2, Theorems 8.1 and 8.3. \square

Remark. The statement of Corollary 12.4 also holds for $\alpha = \omega$. This can be seen from the proof of the main result in [9].

We finish with stating two open problems.

- Problems 12.5.**
- a) *Does there exists a family of metric spaces (M_i, d_i) which is a family of test spaces for reflexivity in the sense of [25], i.e., for which it is true that a separable Banach space X is reflexive if and only if not all of the M_i uniformly bi-Lipschitzly embed into X ?*
 - b) *Does there exists a countable family of metric spaces (M_i, d_i) which is a family of test spaces for reflexivity?*
 - c) *It follows from Theorem B, that if X is a separable Banach space with non separable bidual then $(\mathcal{S}_\alpha, d_{1,\alpha})$ bi-Lipschitzly embeds into X , for all $\alpha < \omega_1$. Is the converse true, or in Ostrovskii’s language, are the spaces $(\mathcal{S}_\alpha, d_{1,\alpha})$, $\alpha < \omega_1$ test spaces for spaces with separable bi-duals?*

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